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I

AGGREGATES OF ZERO MEASURE[1]

I

We say that a linear aggregate E is of measure zero if, when we are given a number ϵ arbitrarily small, we can inclose all the points of E within intervals whose sum is less than ϵ. For an aggregate of two dimensions we have a similar definition, replacing the intervals by the rectangles. Moreover, we see that we may speak of squares instead of rectangles, because if we are given a rectangle we can find a finite number of squares of which the total area differs as little as we please from the area of the rectangle, and such that every point within the rectangle is also within one of these squares. We could also replace squares by circles without altering the generality of the definition.

Aggregates of measure zero play a very important part in the theory of functions of a real and of a complex variable. It is therefore useful to be able to compare the different aggregates of measure zero among themselves. This comparison is aided by the concept of regular aggregates. In the first place, then, we shall define regular aggregates and the fundamental points of these aggregates, and we shall show that every regular aggregate is equivalent to another regular aggregate of which the fundamental points are chosen in a special manner, for example, as the points with rational coördinates. Finally, we shall consider the classification of aggregates of measure zero, with given fundamental points. This classification will be based on the asymptotic decrease of the intervals (or squares) of exclusion.

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[1] Translated from the French by Professor Griffith Conrad Evans, of the Rice Institute.

An aggregate of measure zero is said to be regular when it can be defined in the following manner:

Let $A_1, A_2, \cdots, A_n, \cdots$ *be a denumerable infinity of points, said to be fundamental points. To each integral number h let us make correspond an infinity of squares* $C_1^{(h)}, C_2^{(h)}, \cdots, C_n^{(h)}, \cdots$, *of which the areas form a convergent series, such that the square* $C_n^{(h)}$ *incloses in its interior* $C_n^{(h+1)}$ *and approaches* A_n *when h increases indefinitely. Let* E_h *be the aggregate of points inside of the squares* $C_n^{(h)} (n = 1, 2, \cdots)$. *The aggregate of points contained in all the* E_h *is a regular aggregate* (which is evidently of zero measure).

Every aggregate of zero measure can be considered as part of a regular aggregate. In other words, if A is any aggregate of measure zero, we can define a regular aggregate E of zero measure, such that every point of A belongs to E. To prove this proposition let us imagine a sequence of numbers $\epsilon_1, \epsilon_2, \cdots, \epsilon_n$, decreasing and tending to zero, the series $\Sigma\epsilon_n$ being supposed convergent. Since the aggregate A is of measure zero, we can define an aggregate $A^{(h)}$ of squares (with sides parallel to the axes) the sum of whose area is less than ϵ_h, and such that every point of A is inside one of these squares $A^{(h)}$. We define first the squares $A^{(1)}$, then the squares $A^{(2)}$; if there are portions of these squares $A^{(2)}$ which are outside all the squares $A^{(1)}$, we can suppress them as useless. In order to proceed in a perfectly definite manner, we consider the first of the squares $A^{(1)}$, say $A_1^{(1)}$, and operate successively on the portions of the successive squares $A^{(2)}$ which are inside $A_1^{(1)}$; we continue in the same way with $A_2^{(1)}$, being careful each time to omit the portions already considered, etc. These operations lead us to consider rectangles, each of which may be replaced by an enumerable infinity of squares (in particular cases a finite number). It is sufficient, in order to form the squares according to a definite law, to construct successively the greatest possible square inside the rectangle, taking as the vertex nearest the origin of coördinates that vertex of the rectangle which is nearest the origin of coördinates. If among the squares so defined there are some which contain no point of the aggregate A we suppress them. We may assume the squares to be arranged in the order of decreasing size (if two of them happen to be equal in size we shall arrange them according to the relative values of the abscissas of their centers; and if these abscissas are equal, according to the value of their ordinates). In the same way we arrange the squares $A^{(2)}$ (after the required transformations), and so on.

We define an aggregate B of squares which will consist of all the squares $A^{(1)}$, and besides a certain number of the squares $A^{(2)}, A^{(3)}, \cdots$. In the same way $B^{(2)}$ will include all the squares $A^{(2)}$ and, besides, a certain number of the squares $A^{(3)}, \cdots$. It is clear that the sum of the squares $B^{(h)}$ is less than $E_h + E_{h+1} + \cdots$ is finite no matter what h may be and approaches zero when h increases indefinitely. Since all the squares $A^{(h)}$ will be part of the $B^{(h)}$, every point of A is inside of one of the squares $B^{(h)}$. In order that the aggregate E defined by the $B^{(h)}$ may be regular we must be able to number the $B^{(h)}, B_1^{(h)}, B_2^{(h)}, \cdots, B_n^{(h)}, \cdots$, in such a way that $B_n^{(h+1)}$ shall be less than $B_n^{(h)}$.

We achieve this result in the following manner. Consider first the squares $A^{(1)}$, if there are any, whose area is greater than ϵ_2 (we know that there are none whose area is greater than ϵ_1, since the sum of all the $A^{(1)}$ is less than ϵ_1). We designate these squares as $B_1^{(1)}, B_2^{(1)}, \cdots, B_{n_2}^{(1)}$. Let us consider next those remaining squares $A^{(1)}$ of which the area is greater than ϵ_3, and let us denote them by $B_{n_2+1}, B_{n_2+2}, \cdots, B_{p_2}^{(1)}$. Let us take now the squares $A^{(2)}$ whose area is greater than ϵ_3; they are arranged in a definite order, as we have said. If the first of them is inside one of the $A^{(1)}$ already numbered, for example inside $B^{(1)}_k$, we shall denote it by $B^{(2)}_k$, otherwise we shall denote it at the same time by $B^{(1)}_{p_2+1}$ and by $B^{(2)}_{p_2+1}$. In the same way, if the second of the $A^{(2)}$ that we take is inside one of the $A^{(1)}$ already numbered, different from $B^{(1)}_k$, say $B^{(1)}_h$, we shall denote it by $B^{(2)}_h$. If it is not inside any of the $A^{(1)}$ (it cannot be inside an $A^{(1)}$ without a number, since its area is greater than ϵ_3 and the $A^{(1)}$ without numbers have areas less than ϵ_2), or if it is inside the particular $B^{(1)}_k$ which has already been utilized, we shall denote it at the same time by $B^{(1)}_{p_2+2}$ and by $B^{(2)}_{p_2+2}$. In this way we manage to define a certain number of new squares $B^{(1)}$ which we will call $B^{(1)}_{p_2+1}$, $B^{(2)}_{p_2+2}$, $\cdots$, $B^{(1)}_{n_3}$, and a certain number of squares $B^{(2)}$ which include all the $A^{(2)}$ of area greater than ϵ_3.

Let us consider now the squares $A^{(1)}$ of area greater than ϵ_4, and let us denote them by $B^{(1)}_{n_3+1}$, $B^{(1)}_{n_3+2}$, $\cdots$, $B^{(1)}_{p_3}$; we can proceed in the same way as before for the $A^{(2)}$ whose areas are greater than ϵ_4, and we can then pass on to the $A^{(3)}$ whose areas are greater than ϵ. Those among them which are inside of the $B^{(2)}$ already numbered will have the same numbers (each number being given of course but one time). The others will be denoted at the same time by $B^{(1)}_3$, $B^{(2)}_3$, $B^{(3)}_3$. We can continue indefinitely in the same way, the ϵ_k approaching zero when k increases indefinitely and each operation involving only a finite number of squares. In this way every square belonging to $A^{(h)}$ will appear in $B^{(h)}$ in a determinate position. Moreover, it is obvious that $B^{(h)}_q$ approaches zero no matter what q may be when h increases indefinitely. It is impossible that certain series $B^{(1)}_q$, $B^{(2)}_q$, $\cdots$, $B^{(r)}_q$ should terminate, because that would mean that no one of the squares $A^{(r+1)}$ is inside $B^{(r)}_q$; that is to say, that $B^{(r)}_q$ would inclose no point of the aggregate A, which is contrary to our hypothesis. The aggregates of squares $B^{(h)}$ define, then, a regular aggregate which includes all the points of A, and our theorem is proved.

We notice that in the definition of the regular aggregate E there are certain series $B_q^{(1)}, B_q^{(2)}, \cdots$, of which a certain number of the first terms denote squares that coincide among themselves. That, in fact, is no difficulty. We can, however, avoid this circumstance by slightly modifying the definitions of the first B_q of such a series; if, for instance, $B_q^{(1)}$, $B_q^{(2)}$, $B_q^{(3)}$ coincide, we can replace $B_q^{(2)}$ by $(1+\epsilon)B_q^{(2)}$, and $B_q^{(1)}$ by $(1+\epsilon_1)(1+\epsilon)B_q^{(1)}$ (we designate by αC a square similarly placed to C, with the ratio α of similarity). These operations multiply the total extent of the squares $B^{(h)}$ by a factor less than the convergent infinite product $\Pi(1+\epsilon_k)$.

We notice that the regular aggregate E which we have defined is not necessarily the most simple of the regular aggregates of measure zero which include the A, but it is not important that our demonstration should give us *the most simple*. The essential thing is to show that there exists *one;* it is then possible to consider without contradiction the collection of all the regular aggregates of measure zero which contain A, and we can choose from this collection if not the simplest (which may not exist, in the same way that the smallest number greater than $\sqrt{2}$ does not exist), at least an aggregate E whose simplicity is as close as we please to the greatest possible.

From now on we shall consider especially the regular aggregates. Such an aggregate is defined by the fundamental points A_n, which are limits of the $B_n^{(h)}$ when h increases indefinitely, and by the magnitudes of the excluding squares $B_n^{(h)}$ corresponding to A_n.[1] The derived aggregate of the fundamental points is a closed set A'. In the general case this set is composed of a perfect aggregate and a reducible aggregate. The excluding intervals which correspond to the points of the reducible aggregate have only in common the points of this reducible aggregate itself. Their study therefore gives us nothing new. The really interesting part of a regular aggregate of zero measure is that which is attached to those points of A' which form a perfect aggregate. We shall have to distinguish cases according to the nature of this perfect aggregate. We shall limit ourselves, however, to the consideration of the case where the aggregate A' contains all the points of a certain area of simple form. The points A_n will then be dense within this area.[1] All the cases where the area is of a single piece and simply connected may be reduced by conformal representation to the case of the area bounded by a circle. We shall show that if we have two different systems of points A_n and B_n, dense within the interior of equal circles and also dense on their circumferences,[2] we can establish between these points a reciprocal continuous one-to-one correspondence in such a way that the ratio between the distance of any two points A_p, A_q and the distance of the corresponding points B_p, B_q will be included between two limits as close to unity as we please. It will follow from this theorem that we shall be able without loss of generality to suppose that the fundamental points of an aggregate of zero measure, when these points are dense within a certain region, coincide with a given dense aggregate in that region — for instance, with the points of rational coördinates.

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[1] It might seem desirable to consider also the relative positions of the A_n in these squares; but by modifying slightly the definitions we can so arrange that every $B_n^{(h)}$ has A_n for its center.

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[1] We shall thus leave aside those aggregates of zero measure which we obtain by assuming that A is a perfect linear aggregate which without being linear yet contains no area. For example, we could exclude certain fixed areas around the points with rational coördinates and take for the A_n the points with algebraic coördinates which did not belong to the excluded areas. We could also build up in some arrangement several similar constructions, or even a denumerable infinity of such constructions superposed, and thus obtain regions which would be quite complicated from the point of view of Analysis Situs.

[2] The case when neither aggregate has points on the circumference can be treated in the same way.

II

The theorem which we have in view can be expressed as follows: *Given two equal circles C and C′, and two enumerable aggregates A and B, of which the first is dense in C and on the circumference C, and the second is dense in C′ and on the circumference C′, and given an arbitrarily small number ε, then we can number the points of A and B in such a way that to a point on the contour we make correspond a point on the contour, and that we have, whatever p and q may be,*

$$1 - \epsilon < \frac{A_p A_q}{B_p B_q} < 1 + \epsilon.$$

We shall say that in this case the two aggregates are *similar by* ε.

In order to prove this theorem we shall assume that the points of the two aggregates are arranged provisionally in a determinate order, and we shall consider successively the first point of A, then the first point of B, then the second point of A, then the second point of B, and so on. Thus we shall not miss any point belonging to either of the two aggregates. To each new point that we consider in one aggregate, we shall make correspond a determinate point in the other; and when the turn of this new point comes we shall omit it.

We shall suppose that the centers of the circles C and C' do not belong to the aggregates A and B (nothing would be changed if both of them should belong, for we could make them correspond; and if one of them belonged, but the other not, we could make a conformal transformation, differing little from the identical transformation, which would transform the second circle into an equal circle whose center could then be made to correspond to the center of the first circle). In this way we can investigate the two circles by considering them superposed and yet distinct. It is possible now to choose two rectangular axes Ox and Oy in such a way that the diameters parallel to the axes contain no points of A or B, and every line parallel to either of the axes contains at most one point of A and one point of B (because the totality of directions of lines which connect the center with points of A or with points of B, or connect the points of A among themselves, or the points of B among themselves, or which are perpendicular to these directions, form an enumerable aggregate). Let us assume an infinite series of positive numbers $\epsilon_1, \epsilon_2, \cdots, \epsilon_n, \cdots$ such that

$$1 - \epsilon < \Pi(1 - \epsilon_n) \quad , \quad \Pi(1 + \epsilon_n) < 1 + \epsilon.$$

The circle C is divided by the diameters parallel to the axes in four equal regions which provisionally we shall call 1, 2, 3, 4; and the circle C' is divided in homologous regions which we shall designate in the same manner.

Consider first A_1, which may be, for instance, in the region 3: since it cannot be on the diameters, it must be *inside* this region, unless it be on the circumference (a case which we are for the moment excluding). Let us now designate by A'_1 the point of the region 3 of C' which coincides with A_1 when C' is moved upon C by a translation. If A'_1 happens to belong to B, which is not the general case, we shall call it B_1. Otherwise we shall define a square with center at A_1, such that the ratio of the greatest to the least of the shortest possible distances of all the points in the square to points on the boundary of the region 3 shall be less than $1 + \epsilon_1$. This shortest distance is parallel to the axes for the rectilinear portions of the boundary and coincides with the radius for the curvilinear portion, and, from our hypothesis in regard to A_1, is not zero. So the construction of the square is always possible. We now choose B_1 arbitrarily from the points of

B inside this square (if we wish to avoid having to make an arbitrary choice from among a denumerable infinity of points, we can take the point of B whose number is smallest in the provisional classification). Having chosen this point B_1 we construct parallels to the axes passing through A_1 and B_1, each set of which, with the diameters already drawn, will divide its circle into regions (nine in each) which will correspond two by two. Some of these regions will be rectangles (in this case only one), while the others will be quadrilaterals or triangles of which certain sides are parallels to the axes and one side is an arc of the circle. If we agree to consider as the dimensions of such regions the dimensions of the rectilinear sides, it follows from the construction that the ratio between homologous dimensions of two corresponding regions is included between $1 + \epsilon$ and $1 - \epsilon$.[1] In the case which we have momentarily excepted, where A_1 is on the circumference, we can take B_1, also on the circumference, in such a way that the same condition shall be verified with respect to the regions, a construction which is always possible.

Let us turn now to the second point B_2, taken from the second aggregate. We make correspond to it a point A_2 situated in the homologous region, chosen in such a way that the new regions obtained by drawing parallels to the axes through A_2 and B_2 have homologous sides whose dimensions are included between $(1 + \epsilon_1)(1 + \epsilon_2)$ and $(1 - \epsilon_1)(1 - \epsilon_2)$. This condition necessitates assigning to A_2 a certain area inside this region, and A_2 is chosen inside this region either arbitrarily, or according to some definite law, as has been explained for B_1, care being taken to have A_2 on the circumference C, if B_2 is on the circumference C'.

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[1] We have, in fact, $\frac{1}{1 + \epsilon_1} > 1 - \epsilon_1$, and, according to our construction, the ratios of homologous sides are included between $1 + \epsilon_1$ and $\frac{1}{1 + \epsilon_1}$.

We continue in the same way, taking alternately a point in A and a point in B, making it correspond to some point in the other aggregate. After n operations we shall have at most $(n+2)^2$ regions, and the ratio of two homologous dimensions of two regions which correspond will always be included between

$$(1-\epsilon_1)(1-\epsilon_2)\cdots(1-\epsilon_n)$$

and

$$(1+\epsilon_1)(1+\epsilon_2)\cdots(1+\epsilon_n)$$

and therefore between $1-\epsilon$ and $1+\epsilon$. If we continue in this way indefinitely, every point of A and every point of B will have a number, after a finite number of operations, and this number will be at most double the number of the same point in the provisional classification.

This final classification satisfies completely the conditions of our theorem. For, if we consider any two points A_p, A_q, with their corresponding points B_p, B_q, the difference of the abscissa x_p and x_q of A_p and A_q, when the regional division has progressed far enough (that is, after a number of operations not greater than the larger of the two members p, q), will be equal to the sum of the rectilinear sides of certain regions, and the abscissas x'_p, x'_q of B_p and B_q will be equal to the sum of rectilinear sides of the corresponding regions. We shall have then

(1) $$1-\epsilon<\frac{x'_p-x'_q}{x_p-x_q}<1+\epsilon$$

and similarly

(2) $$1-\epsilon<\frac{y'_p-y'_q}{y_p-y_q}<1+\epsilon,$$

from which follows immediately

$$1-\epsilon<\frac{\sqrt{(x'_p-x'_q)^2+(y'_p-y'_q)^2}}{\sqrt{(x_p-x_q)^2+(y_p-y_q)^2}}<1+\epsilon.$$

But this last relation is the statement of our theorem.

We might show in the same way the analogous theorem about the angles α and β which the lines A_pA_q and B_pB_q make with the axis Ox. In fact, we have

$$\tan\alpha = \frac{y_p - y_q}{x_p - x_q}, \quad \tan\beta = \frac{y'_p - y'_q}{x'_p - x'_q},$$

so that from equations (1) and (2) we deduce immediately

$$\frac{1-\epsilon}{1+\epsilon} < \frac{\tan\alpha}{\tan\beta} < \frac{1+\epsilon}{1-\epsilon}.$$

If we take the angles α and β positive, since they are almost of the same value, $\cot\beta + \tan\alpha$ is greater than or at least equal to 2, and therefore, neglecting ϵ^2, we shall have

$$|\alpha-\beta| < |\tan(\alpha-\beta)| = \left|\frac{\dfrac{\tan\alpha}{\tan\beta} - 1}{\dfrac{1}{\tan\beta} + \tan\alpha}\right| < \frac{1}{2}\left|\frac{\tan\alpha}{\tan\beta} - 1\right| < \epsilon.$$

The properties of the correspondence which we have shown to exist between two enumerable aggregates A and B, which are dense in equal circles C and C', are worth studying more completely. Here follow some remarks that might be useful in such a study. In the first place we observe that if any partial arrangement of points A_{n_1}, A_{n_2}, $\cdots$ approach a limiting point P, there corresponds to it a partial series of points B_{n_1}, B_{n_2}, $\cdots$ which approaches a limit P'. The correspondence between P and P' is well defined, — that is, is independent of the partial series that may be chosen. We have in this way a one-to-one correspondence between the points of C and the points of C'.

Let us agree to call the parallels to the axes, drawn through the points of the aggregate, lines of discontinuity. To any point M not on a line of discontinuity corresponds an homologous point M', and the transformation of the region in the neighborhood of M into the region in the neighborhood of M' may be written in the form

$$\begin{aligned} x' &= (h + \eta)x \\ y' &= (h + \eta')y, \end{aligned}$$

where x, y are the coördinates of M, x', y' are the coördinates of M', h, k are constants of value between $1 - \epsilon$ and $1 + \epsilon$, and η and η' are functions of x and y which approach zero when $x^2 + y^2$ approaches zero. The constants h and k are the two ratios of similitude (parallel to the two axes) of the neighborhoods of M' and M. If the points M' and M lie on a line of discontinuity, the ratio of similitude in the direction perpendicular to this line has not the same value on both sides of the line. At a point M which is the intersection of two lines of discontinuity, there are four values for each ratio of similitude, corresponding respectively to the positive and negative variations of the two coördinates. The ratio of similitude h is thus defined throughout C. It is discontinuous on the lines of discontinuity, but continuous at other points.

If we know nothing about the provisional numbering of the aggregates A and B, we can merely say this about the relation between the provisional numbering and the final numbering: that the final number n is at most twice the provisional number p; for every point numbered provisionally A_p or B_p is chosen after at most $2p$ operations. We cannot, however, give an upper limit to p as a function of n.

It will be possible to determine such a limit, provided that we take care to choose the system of provisional numbering from among those that are *sensibly homogeneous.* Let us make our meaning clear. By definition, in order to arrange a very large number p of points in a homogeneous manner in a circle C, we shall construct a square grating such that p of its vertices are inside C; if α_p is the length of a segment of the grating, we shall put one point in each square of side α_p, and l^2 in each square of $l\alpha_p$, exactly if l is an integral number, approximately if l is not. Let us write $l\alpha_p = \lambda$ and take λ as fixed and p variable. Then for every value of p we can calculate the approximate number of points inside the square of side λ, a number which may be given asymptotically as $p\lambda^2/\pi r^2$, r being the radius of the circle C. We shall say that the arrangement of points of the enumerable aggregate $A_1, A_2, \cdots, A_p, \cdots$ is asymptotically homogeneous if, for any square of side λ, the number λ_p of points of index less than p inside this square approaches this same symptotic value $p\lambda^2/\pi r^2$ when p increases indefinitely; *i.e.*, if the ratio $\pi\lambda_p r^2/p\lambda^2$ between the numbers λ_p and the symptotic value $p\lambda^2/\pi r^2$ approaches 1 as p increases indefinitely. We shall say that the arrangement is *sensibly homogeneous* if this ratio becomes and remains limited by two constants α and $\beta (\alpha < 1 < \beta)$ independent of p and of the position of the square of side λ.

In the preceding definition of homogeneous arrangement, nothing was said about the points that happened to be situated on the boundary. If the boundary is a square of side a, the maximum number of points situated on this boundary for a grating of measure a/n is $4n$, the total number of points being n^2. Generally speaking, the number of points on the boundary will be said to be normal if it is of the order of magnitude of the square root of the total number of points. We must observe that this notion of normal depends on the assumption that *there are* points on the contour. If the points were arranged *arbitrarily*, in the general case there would be *no point* on the boundary, and this is indeed the simpler hypothesis. But if *there are* points on the contour, the case is probably that there is some sort of a relation between the way the contour is chosen and the way the points are given. Hence it is natural to suppose that the probability that a point falls on an arc of the boundary of unit length is some finite proportion of the probability that a point falls in unit area. This hypothesis is verified, for instance, if the boundary is a circle and if the points of the aggregate are those with rational coördinates. Other such hypotheses might be conceived, related to the theory of numbers.

We must then, in the case where there are points on the boundary, add to the hypothesis that the arrangement is sensibly homogeneous inside, the hypothesis that it is sensibly homogeneous on the boundary.

In many questions, the preceding definition of sensibly homogeneous arrangements is inadequate; it is necessary to add a condition which may be called *intrinsic homogeneity*, because it introduces the relative positions of the points of the aggregate. If we consider the vertices of a grating, which we take as the type of homogeneity (or, say, a net of equilateral triangles), we see that the shortest distance between two vertices is proportional to the inverse square root of the total number of points. We say that a two-dimensional aggregate is intrinsically homogeneous if the shortest distance between any two of its points of number less than p is of the order of magnitude $1/\sqrt{p}$.[1] Homogeneity of arrangement and intrinsic homogeneity are thus seen to be independent conceptions, neither being a consequence of the other.

Given a denumerable aggregate, dense within a circle (or square), it is always possible to number its points in such a way as to satisfy the conditions of homogeneity. One of the simplest methods of doing this is as follows. After having numbered some of the points, we trace a grating fine enough to make a few more squares than points already numbered, and such that one square includes at most one of these points. There will then be some squares that do not contain such points. In each of these we number one point of the aggregate, by choosing it inside a square concentric with the first, and twice smaller, taking the point of smallest subscript in the provisional numbering (thus we are sure of not omitting any point).

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[1] An analogous condition should be verified for the shortest distance to the boundary of points very near to this boundary and not lying on it.

Any system of numbering that satisfies both conditions of homogeneity will be spoken of as *normal.* It is easy to verify the fact that the methods of numbering habitually used lead to normal arrangements.

When the two aggregates that are dense in C and C' are numbered normally, it is possible to arrange matters so that the one-to-one correspondence set up between their elements shall be itself normal; *i.e.*, there exist between the provisional numbering, p, and the final numbering, n, inequalities of the form

$$p^{\alpha} < n < p^{\beta},$$

where the exponents α and β are finite and depend only on the number of dimensions in the aggregate considered, and on the convergent series $\Sigma \epsilon_n$ which has been used. (In order to be sure that α and β are finite, there must be a finite quantity h such that $\lim n^h \epsilon_n = 0$.)

We divide the aggregate A into two others, A' and A'', still everywhere dense, and the aggregate B, similarly, into B' and B''. It is then easy to show that the correspondence can be set up in such a way that the points of A' correspond to those of B' and the points of A'' to the points of B''. For that, it would not be sufficient of course to apply the general theorem first to A' and B' and then to A and B, because the correspondence thus set up between two points P and P' inside C and C', respectively, would not in general be the same by means of the two separate correspondences.

This procedure we can extend to the case where A and B each consists of a denumerable infinity of aliquot parts, everywhere dense. We can establish, for instance, a continuous one-to-one correspondence between the rational numbers in a certain interval, and the algebraic numbers in an equal interval, in such a way that to the rational numbers whose denominators consist of h and only h distinct prime factors, correspond the algebraic numbers which are the roots of an irreducible equation of degree h (for $h = 1$ we get the rational numbers; if we wish to consider only the irrational algebraic numbers we must take irreducible equations of degree $h + 1$).

III

Let us consider now two regular aggregates of zero measure, of which the fundamental points are precisely the denumerable aggregates A and B inside the circles C and C'. If we suppose that the squares of exclusion belonging to the corresponding fundamental points have as their sides lines which correspond, it is evident that the two aggregates will correspond point by point in the one-to-one correspondence that we have established between the points P inside C and the points P' inside C'. In other words, *given a regular aggregate of zero measure of which the fundamental points B are dense in C', we can define a regular aggregate of zero measure of which the fundamental points are the elements of an arbitrary aggregate A, dense in C, in such a way that the two aggregates correspond to each other continuously and in a one-to-one manner* (the ratio of similitude being contained between $1 - \epsilon$ and $1 + \epsilon$).

Hence in order to study regular aggregates of zero measure of which the fundamental points are dense within a certain region, we can without loss of generality assume that the fundamental points are, for instance, the points with rational coördinates. In particular it is easy to prove this important proposition: *Every regular aggregate of zero measure of which the fundamental points are dense within a certain region has the order of the continuum.* In other words, if we arrange at pleasure the diminishing of the squares of exclusion in the neighborhood of fundamental points, it is not possible to make this diminution rapid enough so that the fundamental points shall be the only ones of the aggregate.

For simplification let us consider the case of a single dimension; the demonstration is in principle the same for any number of dimensions. Let A_n be the intervals of exclusion belonging to the points A_n. For each value of h we can define a positive function $\phi_h(n)$ increasing with n, such that we shall have

$$\text{measure } (A_n^h) > \frac{1}{\phi_h(n)}.$$

On the other hand, if we are given a denumerable succession of increasing functions $\phi_h(n)$, it is possible, according to a theorem of Paul du Bois-Reymond, to construct a function $\phi(n)$ increasing more rapidly than any of the functions $\phi_h(n)$. After having found this function $\phi(n)$, the theory of continuous functions enables us to define an infinite number of irrational numbers x (an infinity which has the order of the continuum) such that there exists for each of them a denumerable infinity of relations of the form

$$\left| x - \frac{m}{n} \right| < \frac{1}{\phi_h(n)}$$

where m and n are integers. Such a number x, whatever h may be, belongs to at least one of the intervals $A_n^{(h)}$; it is therefore an element of the aggregate defined by the points

A_n and these intervals of exclusion. In order to define the numbers x and show that their aggregate is of the order of the continuum, it is sufficient to investigate a continuous fraction in which the incomplete quotients increase very rapidly. If we write

$$P_{n+1} = a_n P_n + P_{n-1}$$

$$Q_{n+1} = a_n Q_n + Q_{n-1}$$

and assume that

$$a_n > \phi(Q_n),$$

where $\phi(n)$ is the function which we have just defined, we shall have, from the nature of the convergents,

$$\left| x - \frac{P_n}{Q_n} \right| < \left| \frac{P_{n+1}}{Q_{n+1}} - \frac{P_n}{Q_n} \right| = \frac{1}{Q_n Q_{n+1}} < \frac{1}{\phi(Q_n)}.$$

But the totality of systems of integers a_n which verify the relations $a_n > \phi(Q_n)$ have, themselves, the order of the continuum.[1]

If we wished to have intervals of exclusion which should decrease rapidly enough so that the aggregate of points defined by them would be composed only of the fundamental points, the $\phi_h(n)$ would have to contain functions increasing more rapidly than any $\phi(n)$. According to the theorem of Paul du Bois-Reymond, that is not possible if the indices h are denumerable. It would be necessary then to make belong to any fundamental point a transfinite infinity of intervals of exclusion, the corresponding functions $\phi_\alpha(n)$ (where α denotes a transfinite number) being such that every increasing function $\phi(n)$ is surpassed by one of them. In this way, however, we get outside the domain of definitions expressible in a finite number of words.

In order to classify the regular aggregates of zero measure, it is better to consider rather than the functions $\phi_h(n)$ which we have defined, the functions $\psi_h(n)$ determined by the relations

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[1] Each a_n may be odd or even; the aggregate of x then includes an aggregate of the same order as that of the numbers 0.1010110···, written in the binary scale.

$$\sum_{p=n}^{\infty} \text{measure } A_p^{(h)} = \frac{1}{\psi_h^{(n)}}.$$

The convergence of the series formed by the intervals of exclusion of order h implies that the functions $\psi_h(n)$ should increase indefinitely with n. After the theorem of Paul du Bois-Reymond, there exists a function $\psi(n)$ increasing less rapidly than any of these, which nevertheless approaches $+\infty$ as n approaches ∞. Hence whatever the value of h, if we take n large enough, we shall have

$$\sum_{p=n}^{\infty} \text{measure } A_p^h < \frac{1}{\psi(n)};$$

that is to say, that the different series formed by the intervals of exclusion all converge more rapidly than the series

$$\sum\left[\frac{1}{\psi(n)} - \frac{1}{\psi(n+1)}\right].$$

The more rapid the increase of $\psi(n)$, the fewer points are included in the aggregate of measure zero, because the intervals of exclusion in that case decrease more rapidly. It is natural, then, to take the function $\psi(n)$ as defining what we may call the asymptotic order of the regular aggregate of zero measure. These orders can be expressed by means of the notations used for orders of infinity; $\psi(n) = n^p$ will be said to be of order p, $\psi(n) = \epsilon^n$ of order ω, ϵ^{ϵ^n} of order ω^2, etc. We meet the aggregates of order ω^2 in defining monogenic functions which are not analytic.

IV

Perhaps it is opportune to emphasize a little the general conclusions which follow from this rapid study of aggregates of zero measure.

Aggregates of zero measure have a fundamental position in the theory of functions. It is, in fact, always possible to inclose the singularities of finite functions in aggregates which are either of zero measure or of measure as small as we like. On the other hand, aggregates which are not of zero measure have a uniform quality, being formed of continuous aggregates either positive or negative. They are heterogeneous with regard to the continuum. Aggregates of zero measure can, however, be sensibly homogeneous with regard to the continuum, that is to say, identical with themselves in intervals as small as we like.

The concept of aggregate of zero measure is so general that we cannot hope to make a profound investigation of the properties of functions without studying minutely this general notion. That is to say, we must not regard all aggregates of zero measure as undifferentiable. The classification based on the asymptotic diminution of intervals of exclusion seems to me to be a first step in this study which faces the students of analysis.

With this question, as with all those where the general notion of increasing functions enters (as, for example, in the theory of the convergence of series with positive terms), difficulties of a transfinite nature are presented which we cannot hope entirely to surmount. But, on the other hand, the problems which are actually met with are generally if not always free of these difficulties (this is the case, for instance, with the usual criteria for the convergence of series of positive terms; for, although theoretically quite special, they are nevertheless practically sufficient for the treatment of the series which are presented in all researches in analysis). We can legitimately hope that it will be the same way with the classification of aggregates of measure zero. Theoretically the complexity of this classification surpasses that of the study of series of positive terms, a study which will never be finished; but practically, a relatively restricted number of classes will suffice for the needs of analysis.

In closing I should like to direct attention to a notable consequence of the theorem about the correspondence between two denumerable aggregates which are everywhere dense. It might seem natural, passing from the finite to a denumerable infinitive, to suppose that the positions of equilibrium of the centers of gravity of the molecules of a solid body should form a denumerably dense aggregate. But *a priori* it would seem quite an arbitrary hypothesis to suppose that they should coincide with the points of rational coördinates. This simple arithmetic determination seems to have nothing to do with the physical conception. In fact, it evidently is not necessary. But it is as general as any other. The important point is that the hypothesis verifies the conditions of homogeneity of arrangement and intrinsic homogeneity, as we have stated them. The arithmetic treatment of the approximation of numbers by rational numbers is thus the reflection of the general properties of dense aggregates.

II

MONOGENIC UNIFORM NON-ANALYTIC FUNCTIONS[1]

I. THE THEORIES OF CAUCHY, WEIERSTRASS AND RIEMANN

THE integration by d'Alembert of the equation of vibrating strings led to a series of researches out of which the notion of an arbitrary function took shape. Among the geometricians who contributed to clarify the new ideas, there should be mentioned Euler, in the front rank, and besides him Clairaut, Daniel Bernoulli and Lagrange. The question was that of the relation between the analytic and the physical definitions of a function: if a string is displaced *arbitrarily* from its position of equilibrium, does there exist a formula which represents exactly the initial state of the string? Fourier answered in the affirmative and set out the method of calculation of the coefficients of the trigonometric series which represents an arbitrary function. The views put forward by the genius of Fourier have been confirmed by the vigorous analysis of Lejeune-Dirichlet.

The discovery of Fourier revolutionized the notions prevalent up to that time; it was believed, with Euler, that to every analytic expression there corresponded a curve of which successive parts depended on each other: in order to express this interdependence, Euler created the expression 'continuous function': the sense of this expression has since been modified.

Under the influence of the same ideas Lagrange endeavored to prove that every continuous function can be developed in a Taylor series: this series would be the tangible form of the connection, so mysterious till then, between the different arcs of a continuous curve; the knowledge of a small arc would have been sufficient to know the whole curve; but Fourier proved exactly that the problem here was illusory, for the physicist who draws a curve remains at each instant free to modify its aspect; the curve once drawn, it is always possible to represent it in its entirety by a unique analytic expression.

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[1] Translated from the French by Professor Percy John Daniell, of the Rice Institute.

This led to the apparently paradoxical result that there existed no logical reason for regarding two segments of the same straight line, for example, as corresponding to the same function, since it was always permissible likewise to regard as a unique function the ordinate of the continuous curve formed of two different straight lines. At the most it could be said that, in the case of two segments of the same straight line, the formula is simpler than in the case of two segments of different straight lines, but this criterion of simplicity does not seem capable of precise definition, unless one is confined to algebraic functions. The paradox was cleared up by extension of the field of study of functions; Cauchy showed that the properties of real functions could only be well understood if imaginary values of the variable were also studied; the idea of a function of a complex variable became indispensable. Cauchy based this idea on the definition of monogeneity; a function of the complex variable $z = x + iy$ is called monogenic if it has a unique derivative. A function which is monogenic at every point of a region without any exception — that is, not allowing in the region any singular point — can be developed in a Taylor series in the neighborhood of any point in the region; the radius of convergence of the series is equal to the distance from the center to the nearest singular point. From this fundamental theorem

Cauchy deduces the calculation of the integrals of the proper differential equations along any path in the plane.

Cauchy's theory was systematized by Weierstrass and Riemann. Weierstrass defined an *analytic function*, in an exact manner, by means of *elements* and thus arrived at the idea of a region of natural existence, an idea which was contained implicitly in Cauchy's work, but which was not mentioned explicitly by him. Riemann conceived a monogenic function *a priori* independently of any analytic expression and showed the advantages of this geometrical conception.

In reality, the analytic point of view of Weierstrass and the geometric one of Riemann find their most perfect synthesis in Cauchy's fundamental theorem: monogeneity within a circle involves the existence of a Taylor series convergent within the circle. This theorem established a necessary connection between values of the same function as a simple consequence of monogeneity: it is sufficient to know that a function is monogenic within a circle, in order that its value at any interior point should be known by a knowledge of its values in the neighborhood of another point. Since our aim is to define monogenic functions in regions more general than those considered up to the present in the theory of analytic functions, it is necessary to make precise the definition of these new regions.

I shall call a region in which an analytic function can be defined in the sense of Weierstrass a *Weierstrassian region* or *W region*. I shall call regions more general than *W* regions, in which a uniform monogenic function can be defined, *Cauchy regions*, or *C regions*, in honor of the creator of the theory of monogenic functions. We shall see that the essential properties of monogenic functions in the *C* domains which we define are the same as in *W* regions; this does not exclude the possibility of defining C' regions more general again than our C regions. In other words, we cannot assert that our generalization covers all uniform monogenic functions: but it brings us to a definition of a more general class than the class W of the analytic functions of Weierstrass.

W regions are characterized by the following properties. Let us call a Γ circle every circle such that all the points within Γ belong to W. Every point P of W is within a Γ circle: the Γ circles corresponding to two points P and Q of W can be reunited by a finite number of Γ circles cutting each other two by two. To every uniform analytic function there corresponds a W region; inversely, M. Runge has shown that to every W region corresponds an infinity of uniform analytic functions having precisely W as the region of existence.

If it is assumed that there is no other process of analytic continuation than the Taylor series, the boundary of the W region is a *natural limit* of existence of the analytic function, and those portions of the plane, if such exist, which do not belong to W ought to be considered as a lacunar space. On this point Weierstrass has insisted several times, and it has been made conspicuous in the clearest way by M. Henri Poincaré. Let us consider a region D of simple form, such as the interior of a circle, and let us define a function $G(z)$ having D as its lacunar space and another function $G_1(z)$ defined only within D and having consequently all the rest of the plane as its lacunar space. Let us divide the contour of D into two arcs D' and D''. M. Poincaré shows that it is possible to find two uniform functions $F(z)$ and $F_1(z)$ existing in the whole plane, except for the singular line D' for F and D'' for F_1, and in such a way that

$$F + F_1 = G \quad \text{outside } D,$$
$$F + F_1 = G_1 \quad \text{within } D.$$

If then the functions F and F_1 are regarded as uniform, the function $G(z)$ has the continuation $G_1(z)$ which has been chosen entirely arbitrarily; it is then proper to discard all ideas of a continuation within the lacunar space. This paradox is apparently cleared up if it is observed that, when a function such as $F(z)$ possesses a singular line D', supposed impassable, this function remains uniform in Weierstrass's sense when there is added a non-uniform function such as $\log \frac{z - z_0}{z - z_1}$, z_0 and z_1 being two points of the line D'. The remarkable result due to M. Poincaré can then be interpreted by the hypothesis that $F(z)$ and $F_1(z)$ are not really uniform: but in order that this hypothesis should have a meaning, it is necessary to generalize the definition of continuation, in a way so as to be able to pass in certain cases the impassable cuts of Weierstrass; we shall see very soon how this result can be obtained.

But I wished before now to say some words concerning the ideas of Riemann, although it is specially in the study of non-uniform functions, of which I shall not speak here, that Riemann's theory has shown itself productive.

Cauchy has insisted several times on the importance of monogeneity. If an elementary function obtained by a simple calculation made on z is considered and if, for such a function $G(z)$, the ratio

$$\frac{G(z+\delta z) - G(z)}{\delta z}$$

is calculated, this ratio tends to a determinate limit when δz tends to zero, with any argument. Cauchy expresses this essential fact by calling the function monogenic.

If we put

$$G(z) = P(x, y) + iQ(x, y),$$

the condition of monogeneity is translated into the two fundamental equations

$$\frac{\partial P}{\partial x}=\frac{\partial Q}{\partial y},$$

$$\frac{\partial P}{\partial y}=-\frac{\partial Q}{\partial x}.$$

Cauchy has shown that these equations, when they are verified in a region of the plane, involve the existence of the Taylor series; that is to say, of that which can be called analyticity in Weierstrass's sense. Cauchy's demonstration assumes the continuity of the derivative; M. Goursat, in a well-known piece of work, has shown that the existence of the first derivative is sufficient, and involves the continuity and existence of all the derivatives; M. Paul Montel has extended this result to cases where the existence of the derivative has not been assumed in a set of points of measure zero. The statement of these researches is outside my scope; I should mention them nevertheless, because they are in a way complementary to the results which I shall state further on. What is sufficient to remember is that, in the W regions, monogenic functions are analytic; for this reason the expression *monogenic function* is no longer in use by certain geometricians, the expression *analytic function* being considered equivalent; as our aim is precisely to define monogenic functions which are not analytic, it is important to distinguish clearly between the two expressions.

It is difficult to find out if Cauchy conceived the existence of a monogenic function independently of any analytic expression. In fact, he always reasoned about functions which were defined, implicitly or explicitly through known functions, by means of ordinary or partial differential equations; but his reasoning applies without modification to a function defined in a purely ideal way as a correspondence between z and $G(z)$. This was the conception of Riemann, and has certainly rendered good service, as much in the field of real variables where it was introduced by Dirichlet, as in the field of complex variables, by accustoming mathematicians to very general methods of reasoning, made once for all and susceptible of application to cases not foreseen at the time when the reasoning was done. In fact, there is no real difference between Cauchy's and Riemann's point of view; to apply considerations like those of Riemann to *one* determinate function, this function must be defined, that is to say must be distinguishable from other functions; and if this definition is effective, it returns to the category of those which Cauchy admitted. This point belongs to the controversies concerning the axiom of Zermelo; Riemann's point of view is otherwise legitimate, whatever attitude is adopted in this controversy; for those who require a precise definition, it saves one from thinking of all the processes of definition which can be imagined; for those to whom an ideal definition is sufficient, it allows one to treat ideally even those functions which will never be defined practically.

It is by means of Cauchy's fundamental theorem

$$f(\delta) = \frac{1}{2\pi i}\int_c \frac{f(z)dz}{z-\zeta}$$

that it can be shown that monogeneity in a W region involves analyticity in the region. We shall use this theorem also in studying monogenic functions in a region, not W; it will be convenient in order to argue in a general manner about all the possible methods of definition of these functions, to consider them as defined in Riemann's way; that is, to assume that nothing is known about such a function except that it is monogenic. It is necessary to show afterward that a theory thus constructed is not empty, by giving actual examples of functions defined no longer ideally, but explicitly.

I shall restrict myself to stating the definition of C regions in a particular case; if the properties of sets of zero measure studied in the previous lecture are used, it can be seen that this particular statement can be considerably generalized.

Let us consider a W region, and a region within W in which we define a denumerable infinity of fundamental points, everywhere dense; we shall assume that these fundamental points A_n are the points within the circle $|z| = 1$ whose coördinates are rational. To each point A_n is attached a positive number r_n, and we shall assume that these numbers r_n tend very rapidly to zero as n increases indefinitely; we shall define later the manner of decrease; it is sufficient here to know that the remainder of the convergent series $r_1 + r_2 + \cdots + r_n + \cdots$ is less than a quarter of the last term retained; we shall denote by C_q the region obtained by excluding from the W region the points within circles $C_n^{(q)}$ defined as follows. Let us consider circles $S_n^{(q)}$ having as their centers the points A_n and for their radii $\frac{r_n}{2}$; the circle $C_1^{(q)}$ has its center at A_1, and its radius is the smallest of the numbers between $r_1/2^q$ and $r_1/2^{q+1}$ and such that it does not cut any of the circles $S_n^{(q)} (n > 1)$; this is possible in virtue of the hypothesis $r_1 > 4\sum_{2}^{\infty} r_k$, from which it follows that $\frac{r_1}{2^q} - \frac{r_1}{2^{q+1}} > 2\sum_{2}^{\infty} \frac{r_k}{2^q}$; the $S_n^{(q)}$ circles are then either inside $C_1^{(q)}$ (including those which touch internally), or outside $C_1^{(q)}$ (including those which touch externally). We shall take no account of the interior circles, and we shall denote by A_{n_2} the fundamental point of smallest index corresponding to the exterior circles; the circle $C_2^{(q)}$ will have its center at A_{n_2} and its radius the smallest number contained between $\frac{r_{n_2}}{2^q}$ and $\frac{r_{n_2}}{2^{q+1}}$ such that it does not cut any of the circles $S_n^{(q)} (n > n_2)$; it is exterior to the circle $C_1^{(q)}$ since it is interior to the circle $S_2^{(q)}$, and at the same time exterior to the circles $S_n^{(q)}$ of index less than n_2, for these circles are interior to $C_1^{(q)}$ because of the method by which n_2 was chosen. Similarly the circle $C_3^{(q)}$ etc. is defined and one sees that if the region obtained by excluding the points inside circles $C_n^{(q)}$ is denoted by C_q, and the region obtained by excluding the points inside circles $S_n^{(q)}$ by C'_q, all the points of C_q belong to C'_{q+1}, while all the points of C'_{q+1} belong to C_{q+1}; the consideration of the regions C_q is then equivalent to that of the regions C'_{q+1} and evades the difficulties which result from intersections of the circles.

The points of the circumference of $C_n^{(q)}$ are said to constitute the frontier of C_q; the points of C_q which do not belong to this frontier are called *interior* to C_q; it is important to observe that we use the word *interior* here in a different sense from the usual one in the theory of W regions. The points of the set C_q, situated in the interior of the circle of radius 1, form a perfect set, which can be considered as the derived set of the set of its frontier points $C_n^{(q)}$.

The region C is defined as the set of all points such that each of them is interior to some C_q: the region C is not perfect, for it does not contain the points A_n, which are its limiting points. We know that the set (of zero measure) of points which do not belong to C has the power of the continuum. We shall say that a region D is *interior* to C, when all the points of D belong to one and the same C_q, of fixed index. Among the regions interior to C, we shall consider a little more exclusively the regions C_q: every point of C_q is interior to C_{q+1}.

The region C will be said to belong to the class (C) of

Cauchy regions which we are studying here if the numbers r_n are such that, for n sufficiently large,

$$(1) \qquad \log\log\log\frac{1}{r_n} > n;$$

if this condition is verified for two regions C and C', it is verified for the part common to C and C'.

Together with the regions C_p and C, we shall consider *reduced* regions which we shall denote by Γ_p and Γ. To a region C corresponds a determinate system of regions C_p, and an infinity of systems of reduced regions; the following is the definition of one of these systems. Let us suppose numbers ρ_n given, tending to zero rapidly as n increases indefinitely, but much less rapidly than r_n; more precisely, we shall suppose that

$$(2) \qquad \frac{1}{\rho_n^2} < \log\log\frac{1}{r_n};$$

and, at the same time, whatever the fixed number α, that, for n sufficiently large,

$$(3) \qquad \frac{1}{\rho_n} > n^\alpha;$$

these two conditions (2) and (3) are quite consistent by virtue of (1).[1]

The regions Γ_p are defined by means of ρ_n as the C_p's by means of r_n, that is, are limited by circles of radii between $\frac{\rho_n}{2^p}$ and $\frac{\rho_n}{2^{p+1}}$ exterior to each other. The region Γ is formed of the set of points *interior* (in the sense indicated above) to each Γ_p. The regions Γ_p are perfect, Γ is not perfect; the set complementary to Γ has zero measure and the power of the continuum.

The set C contains all points of Γ since C_p contains all points of Γ_p: but C contains besides points which do not belong to Γ.

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[1] (1) (2) and (3) could be replaced by wider conditions: my aim here is to simplify the statement.

The following theorem is fundamental:

If a function of the coördinates of a point P is defined in C and continuous in every C_p, the knowledge of its values at all points of Γ involves the knowledge of its values at all points of C.

In other words, two functions continuous in C (that is, defined in all C and continuous in every region *interior* to C) cannot coincide in all Γ without coinciding in all C; or, finally, *a function continuous in C and zero in Γ is zero in C.*

In fact, let P be a point of C; this point belonging to a set C_p interior to C, it is a limiting point of the set formed by the frontier[1] of C_p; it is sufficient in order to prove that the function is zero at P, since it is continuous in C_p, to show that it is zero on each circumference which constitutes this frontier (the remark has already been made that each of these circumferences is *interior* to C_{p+1}); then, on one of these circumferences (as on every rectifiable curve traced in the plane), the points which are part of Γ are everywhere dense; the function being continuous on this curve is then zero throughout this curve if it is zero at all points of Γ.

When we speak of a reduced region, we shall assume that we consider a determinate region, the ρ_n's being chosen in a precise way, satisfying the inequalities (2) and (3). It might happen that we had to consider at the same time another region Γ' defined by numbers ρ'_n; if

$$p_n'^{\beta}=\rho_n \tag{4}$$

we say that Γ' is of order β with respect to Γ; if β is greater than one, the numbers ρ'_n satisfy the inequalities (2) and (3) when the ρ_n's satisfy them: in this case the set Γ'_p is interior to Γ_p, for the excluded circles of radii $\frac{\rho'}{2^p}$ are larger than the circles of radii $\frac{\rho_n}{2^p}$ (for ρ_n can always be supposed less than 1).

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[1] We neglect points P which would be interior to C in Weierstrass's sense; for them the proposition is evident, since they are centers of circles inclosing no A_n in their interior, they are also interior to Γ in Weierstrass's sense.

Let us remark finally that the points of C_p which lie on any curve whatever, a straight line for instance, form a perfect set, defined by contiguous intervals (in M. Baire's sense), which are the chords intercepted on the straight line by the circles. This set may or may not contain intervals: but in every case it is perfect, and consequently a function continuous in C_p and zero at all the points which limit the contiguous intervals is zero at all points of the set at the same time with all its derivatives in C_p.

II. MONOGENIC FUNCTIONS IN C REGIONS

We shall say that a function $F(z)$ is monogenic in a region such as C if:

1°. It is continuous *within* C (that is, as we have explained, continuous in every C_p, *interior* to C; since the set C_p is perfect, this continuity in C_p is uniform);

2°. At every point P of C, it has a derivative with respect to z, unique and continuous within C. To define the derivative a set C_p of which P is a part is considered, and denoting by P' any other point of C, the limit of the ratio

$$\frac{F(\rho')-F(\rho)}{\overline{\rho\rho'}} \tag{5}$$

is found when the vector $\overline{\rho\rho'} = z' - z$ tends to zero; if this limit exists for every value of ρ, it is evidently independent of the value of p, for all points of C_p belong to C_{p+q}; for this reason this limit can be called the derivative of $F(z)$

within C, that is in every region *interior* to C. The continuity of the derivative within C is to be understood in the same way as the continuity of the function itself *within* C: continuity in each C_p interior to C. This hypothesis of the continuity of the derivative is doubtless superfluous; but it simplifies the argument.

Since the set C_p is perfect, every function continuous in C_p is *bounded* in C_p.

Let us mention at once an example of the simplest kind of a C region and of a function monogenic in this region.[1]

Let us form the series

$$F(z) = \sum_{n=1}^{\infty} \sum_{p=0}^{n} \sum_{q=0}^{n} \frac{e^{-e^{n^4}}}{z - \frac{p+qi}{n}}.$$

Clearly this series is convergent outside the square T of which the vertices are the points $z = 0, 1, i, 1 + i$. Inside this square the series has an infinity of poles; in fact, all the points whose coördinates are rational numbers $x = \frac{p}{n}$, $y = \frac{q}{n}$. But if circles having these poles as centers and radii $\frac{\epsilon}{n^4}$ be considered, the series is absolutely and uniformly convergent at all points outside these circles, whatever the fixed number ϵ may be. The same is true if circles $\Gamma_n^{(h)}$ with centers at the points $\frac{p}{n}, \frac{q}{n}$ and radii $\frac{1}{h} e^{-e^{n^2}}$ are considered, where h is a fixed integer which we are allowed to increase indefinitely. I shall call Γ_h the set of circles $\Gamma_n^{(h)}$ and C_h the set of points which are not inside any of the circles $\Gamma_n^{(h)}$. There exists an infinity of curves which cross the circle and of which all the points belong to one same region C_h.

[1] The region C considered here is a little more general than the regions defined above, in the sense that the series Σ_n^r converges a little less rapidly.

The function $F(z)$ is evidently monogenic within the region C which is the limit of the C_h's; it has in fact at each point of this region a determinate unique derivative, which is obtained by differentiating the series term by term. The value of this derivative is independent of the way in which the increment δz tends to zero, with the reservation, of course, that z and $z + \delta z$ are inside C_h.

The study of monogenic functions within a region C requires the extension of Cauchy's fundamental theory to the contour which limits a perfect region C_h. To this end we shall establish at once the following fundamental property of a function $F(z)$ monogenic in the region C. If we denote by p a fixed number

$$\int_K F(z)dz = \sum_n \int_{C_n^{(p)}} F(z)dz \tag{6}$$

the curve K being any simple curve all of whose points are inside C_p, the sum Σ referring to all the circles $C_n^{(p)}$ which are inside C_p; the integrals are all taken in the direct sense.

We shall set

$$F(z) = P(x, y) + i\,Q(x, y), \tag{7}$$

so that the equation (6) becomes two equations, of which it is sufficient to demonstrate one; for example,

$$\int_K Pdx - Qdy = \sum_n \int_{C_n^{(p)}} Pdx - Qdy. \tag{8}$$

To prove this relation, we define a function $P_1(x, y)$, finite and determinate at all points interior to K, and coinciding with $P(x, y)$ at the points inside K which belong to C_p; there remains the definition of $P_1(x, y)$ inside the circles $S_n^{(p)}$; on the circumference of these circles it coincides with $P(x, y)$. We shall define $P_1(x, y)$ inside the circle by the condition that on chords of the circle parallel to Oy it varies linearly. (Its values at the extremities are known for they coincide with that of $P(x, y)$.) The function $P_1(x, y)$ thus defined is continuous within K and has at every point a derivative $\frac{\delta P_1}{\delta y}$; this derivative is bounded according to the hypothesis that the derivatives of P are bounded (which is involved by the existence and continuity of the derivatives of $F(z)$); in fact, at points inside C_p the derivative of P_1 coincides with the derivative of P; at points inside $C_n^{(p)}$, the derivative of P_1 is constant along a chord parallel to Oy and equal to the quotient of the difference of the values of P_1 (that is, of P) at the ends of the chord, divided by the length of this chord. The difference of the values of P is

$$\int_{MN} \frac{\partial P}{\partial x} dx + \frac{\partial P}{\partial y} dy \tag{9}$$

if MN denotes the arc subtended by the chord.

This integral is less than the product of the length of the arc MN and the sum $\left|\frac{\partial P}{\partial x}\right| + \left|\frac{\partial P}{\partial y}\right|$ and its quotient when it is divided by the chord MN is at most equal to

$$\frac{\pi}{2}\left(\left|\frac{\partial P}{\partial x}\right| + \left|\frac{\partial P}{\partial y}\right|\right)$$

and is consequently bounded at the same time as the derivatives

$$\frac{\partial P}{\partial x} \text{ and } \frac{\partial P}{\partial y}.$$

Similarly the values of P_1 lying between the values of P, P_1 have the same boundary as P.[1]

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[1] The derivative $\frac{\partial P_1}{\partial y}$ is discontinuous at points on a circumference. This produces no inconvenience; one can modify the definition of P_1 by choosing other curves instead of straight lines. Sufficiently simple results can be obtained by taking the sum of a parabola and a sinusoidal curve.

According to a classical result, denoting by $^{(K)}$ the area within K

$$\iint_{(K)} \frac{\partial P_1}{\partial y} dxdy = -\int_K P_1 dx = -\int_K P dx$$

since on K, P_1 coincides with P.

Similarly $Q_1(x, y)$ being defined by means of $Q(x, y)$ in the same way as P_1 by means of P (taking always parallels to Ox in place of parallels to Oy):

$$\iint_{(K)} \frac{\partial Q_1}{\partial x} dxdy = \int_K Q_1 dy = \int_K Q dy$$

It follows that

$$(10) \qquad \int_K P dx - Q dy = -\iint_{(K)} \left(\frac{\partial P_1}{\partial y} + \frac{\partial Q_1}{\partial x} \right) dxdy.$$

The double integral of the right-hand side reduces to zero for those portions of the area (K) which belong to C_p, for at a point inside C_p

$$\frac{\partial P_1}{\partial y} + \frac{\partial Q_1}{\partial x} = \frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x} = 0.$$

The formula (10) then reduces to

$$(11) \qquad \int_K P dx - Q dy = -\sum \int_{C_n}\!\!\int_{(p)} \left(\frac{\partial P_1}{\partial y} + \frac{\partial Q_1}{\partial x} \right) dxdy.$$

But the area of $C_n^{(p)}$ is equal to $\frac{\pi r_n^2}{4^p}$; on the other hand, the moduli of $\frac{\partial P_1}{\partial y}$ and $\frac{\partial Q_1}{\partial x}$ are less than a fixed number independent of n (depending on p, but p is fixed); then

$$(12) \qquad \left| \int_K P dx - Q dy \right| < M \sum r_n^2.$$

It is easy to obtain from this the formula (6); since the series Σr_n^2 is, in fact, convergent, we can choose n in such a way that the remainder of this series $\Sigma_{n+1}^{\infty} r_q^2$ is less than $\frac{\epsilon}{M}$.

When the number n has been thus chosen, let us denote by K' the contour formed of the contour K traversed in the direct sense and the circumferences $C_1^{(p)}, C_2^{(p)}, \cdots, C_n^{(p)}$ traversed in the retrograde sense; we can argue about K' as we have done about K (by completing it if we wish by rectilinear cuts to make it a simple contour); we shall obtain

$$\left|\int_{K'} Pdx - Qdy\right| < M\sum_{n+1}^{\infty} r_q^2 < \epsilon$$

that is, the integrals being taken in the direct sense

$$\left|\int_K Pdn - Qdy - \sum_{r=1}^{n}\int_{C_r^{(p)}} Pdx - \ \ Qdy\right| < \epsilon.$$

If ϵ is made to tend to zero, n increases indefinitely and from it we obtain the relation (8) from which the relation (6) follows.

We deduce now from (6) Cauchy's fundamental theorem; let x denote a point within a reduced region Γ_p and γ_q a circle with center x within Cp, and with radius between $\frac{1}{2^q}$ and $\frac{1}{2^{q-1}}$.

There exists such a circle γ_q, whatever the number q (at least after a certain value of q). In fact x being within Γ_p, whatever n may be, a_n being the affix of A_n

$$| x - a_n | > \frac{1}{2^p} P_n$$

Consequently, the points a_n for which

$$| x - a_n | < \frac{1}{2^{q-1}}$$

are such that

$$\frac{1}{2^p} P_n < \frac{1}{2^{q-1}} \tag{13}$$

Let us denote by n_q the smallest value of n after which this inequality (13) is satisfied; all the a_n's inside the circle of center x and radius $\frac{1}{2^{q-1}}$ have indices greater or equal to n_q; the sum $\frac{1}{2^p}\Sigma r_n$ of the radii of the corresponding circles in C_p is then extremely small compared to $\frac{1}{2^p} P_{n_q}$ since the r_n's are much smaller than the corresponding P_n's; since this sum is extremely small compared to $\frac{1}{2^{q-1}}$, there exist circles of center x and radius between $\frac{1}{2^{q-1}}$ and $\frac{1}{2^q}$ and which do not cut any of the circles $C_p^{(p)}$; *a fortiori* they do not cut the circles $C_n^{(p)}$ whose centers are more distant from x, for the radii $\frac{r_n}{2^p}$ of these other circles are very small compared with $\frac{P_n}{2^p}$ and their centers are further from x than $\frac{P_n}{2^p}$.

The circle γ_q being thus defined, let us consider the function

$$f(z) = \frac{F(z)}{z - x}$$

within the region contained between the contour K and γ_q; clearly in this region this function is monogenic; we then obtain the relation

$$\int_K f(z)dz - \int_\gamma f(z)\,dz = \Sigma \int_{C_n^{(p)}} f(z)dz$$

the sum on the right-hand side referring to the $C_n^{(p)}$'s which are contained between γ_q and K.

If M denotes the maximum value of $|F(z)|$ within C_p the maximum value of $f(z)$ on different $C_n^{(p)}$'s is evidently $2^{q+1}M$; if $q+1$ is put in the place of q, an infinity of new terms are introduced on the right-hand side, but it is easily seen that the lengths of the paths of integration (circumferences of the $C_n^{(p)}$'s contained between γ_q and γ_{q+1}) have a sum of an order much less than $\frac{1}{2^{q+1}}$; the right-hand side is then a convergent series and

$$\lim_{q=\infty}\int_{\gamma_q} f(z)\,dz = \int_K f(z)\,dz - \sum \int_{C_n^{(p)}} f(z)\,dz$$

the sign Σ now referring to all the circumferences C_n^p which limit C_p. As for the left-hand side, it follows from the continuity of $F(z)$ at the point x in C_p, all the γ_q's being interior to C_p, that it is equal to $2\pi i F(x)$. The generalized Cauchy formula follows

$$(14) \qquad 2\pi i F(x) = \int_K \frac{F(z)\,dz}{z-x} - \sum_n \int_{C_n^{(p)}} \frac{F(z)dz}{z-x}.$$

From this formula the classical consequences can be deduced and in particular the fact that *monogeneity (existence of the first derivative) within the region C involves the existence of the derivatives of all orders.* This formula (14) shows moreover that non-analytic monogenic functions can be put in the form of series whose terms are analytic functions. It is natural then to look for an associative method of continuation applicable to such sums. The problem is nothing else than the problem of divergent series: to each analytic function corresponds a Taylor development convergent in a circle, but divergent outside this circle; this development is determined by a knowledge of the values of the derivatives. If a series of analytic functions is indefinitely differentiable, its derivatives are expressed linearly by means of the derivations of the terms, and the Taylor series which corresponds to these derivatives is a linear function of the Taylor series corresponding to the different terms of the series. But if the function is not analytic at the point where the series is developed, this Taylor series will be the sum of series whose radii of convergence decrease indefinitely and, in the case we are studying, will have a zero radius of convergence.

The problem of divergent series consists in transforming such a series into a convergent series in such a way that the result coincides with the analytic continuation in the case where this continuation is possible. Thanks to the fine researches of M. Mittag-Leffler, this problem has been resolved for the first time in an entirely satisfactory way; it should be observed that, if it is desired to use these results for the continuation of non-analytic monogenic functions, they must be interpreted either by the language of divergent series, or by an equivalent language if one prefers not to speak of divergent series; but in every case by a new language, specially adapted to the real novelty of the results, and not by the old language of Weierstrassian analytic continuation; that is the only language which may not be used, since it has an absolutely precise meaning, which cannot be modified; Weierstrass's theory is, in some way, so perfect that it can only be departed from by creating a new language: if, as M. Mittag-Leffler proposed, Weierstrass's language were adopted, M. Mittag-Leffler's series would be only a simplified method of calculation containing nothing more from the theoretical point of view than Weierstrass's theory contains.

III. CONTINUATION BY SERIES (M)

In order to study continuation by M. Mittag-Leffler's series, or series (M), we suppose that the point is interior to a reduced region Γ'_p, of order equal to 2 with respect to Γ_p (the circles of exclusion are defined by numbers ρ'_n equal to $\sqrt{\rho_n}$); evidently then an infinity of straight lines issuing from the point x can be drawn interior to Γ_p. More precisely, if x belongs to Γ', within every given angle having its vertex x, a straight line interior to $\Gamma_{p'}$, of convenient index, can be found; this index can increase indefinitely as the angle tends to zero, but is determinate when the angle is given (this follows from the fact that the sum of the angles subtended at x by the circles which limit $\Gamma_{p'}$ is less than twice the sum of the convergent series $\sum\left(\frac{\rho_n}{2^{p'}} : \frac{\rho'_n}{2^p}\right)$ and is consequently as small as we please if p' is sufficiently large). We shall suppose, so as not to complicate our notation, that p has been taken equal to p' in the preceding argument (the point x interior to Γ_p is *a fortiori* interior to $\Gamma_{p'}$ if $p' > p$).

We develop $F(z)$ in a series on one of the straight lines which we are about to define, interior to Γ_p. Each of the terms of the right-hand side of (14) is an analytic function on this straight line and can therefore be developed in a series of Mittag-Leffler or (M) polynomials; it is enough to show that the multiple series formed of the set of these series is absolutely convergent, in order to show that it represents $2\,\pi i F(z)$.

This series is then formed by means of the derivatives of $F(z)$ at the point x (these derivatives exist, as we have remarked, according to (14) for every displacement on the straight line and in Γ_p), in the same way as the (M) development of an analytic function is formed by means of the derivatives of that function; we assume, to save writing, that $x = 0$.

I remind the reader of the properties of (M) developments which I have demonstrated in my memoir on series of polynomials and rational fractions ("Acta Matematica," I, xxiv). One finds that

$$\frac{1}{1-z} = \sum G_n(z), \tag{15}$$

$G_n(z)$'s being polynomials which it is useless to write again and the series $\Sigma \mid G_n(z) \mid$ being convergent in the 'star.' A region $S(R, \rho)$ is defined as follows: R being > 1 and $\rho < 1$, we consider the circle of center 0 and radius R, the circle of center 1 and radius ρ and the tangents to this last circle from 0, the points of contact being M and N; the region $S(R, \rho)$ is bounded by the arc MN less than π, the continuations MM' and NN' of OM and ON as far as the circumference of radius R and the arc $M'N'$ greater than π. In this region, putting $\frac{32\,R}{\rho} = \lambda$,

$$\sum |G_n(z)| < R^{\lambda^\lambda}. \tag{16}$$

Consider an integral along one of the circumferences $C_n^{(p)}$ of radius $\frac{r_n}{2^p}$.

$$\int_{C_n^{(p)}} f(z)dz \tag{17}$$

We develop it on a straight line interior to Γ_p, that is outside the circumference having the same center a_n as $C_n^{(p)}$ and of radius $\frac{\rho_n}{2^p}$. The radius $\frac{r''}{2^p}$ being very small compared with ρ_n, we shall commit no appreciable error by replacing this integral by the majorant function $\frac{Mr_n}{a_n - x}$, denoting by M the maximum of $|F(z)|$ in C_p, $2\pi r_n$ being the length of the path of integration (we suppress the factors 2^p which have no influence since p is fixed). If one puts $x = a_n x'$,

$$\frac{Mr_n}{a_n - x} = \frac{Mr_n}{a_n} \frac{1}{1 - x'}. \tag{18}$$

If the point x is inside the region $S(R, \rho)$ defined by the circle of radius $\frac{\rho_n}{2^p}$ and center A_n and by a circle of radius > 1 (2 for example) which contains within it all the regions we are considering, the point $x' = \frac{x}{a_n}$ will be within the region

$S\left(\frac{2}{|a_n|}, \frac{\rho_n}{|a_n|}\right)$, and by developing $\frac{1}{1-x'}$ we get the inequality

$$\sum |G_n(x')| < \left[\frac{2}{|a_n|}\right]^{\lambda^\lambda}$$

putting $\lambda = \frac{64}{\rho_n}$; since $|a_n|$ is greater than ρ_n we can write

$$\sum |G_n(x')| < \lambda^{\lambda^\lambda}.$$

The development (M) of (17) is, according to (18), when all the terms are replaced by their moduli, less than

$$\frac{Mr_n}{|a_n|}\lambda^{\lambda^\lambda}$$

But according to (2)

$$\frac{1}{r_n} > e^{e^{\frac{1}{\rho_n^2}}}$$

and if n is large enough $\frac{1}{\rho_n^2} > \lambda^{\frac{3}{2}}$ since $\lambda = \frac{64}{\rho_n}$ and so $|a_n|$ being $> \rho_n$,

$$M\frac{r_n}{|a_n|}\lambda^{\lambda^\lambda} < \lambda M \lambda^{\lambda^\lambda} e^{-e^{\lambda^{\frac{3}{2}}}}.$$

This converges very rapidly to zero when n, and consequently λ, increases indefinitely. The absolute convergence of the (M) series is then demonstrated.

Now consider two points x_1 and x_2 belonging to Γ'; we can construct two angles A_1 and A_2 with vertices at x_1 and x_2, and such that every half-straight line D_1 within A_1 meets every half-straight line D_2 within A_2 at a point x_3 within the total region considered. We can choose D_1 and D_2 in such a way that these two straight lines belong to the same Γ_p (p being chosen large enough, but afterwards remaining fixed). It will then be possible to calculate the function at x_2 by means of its values and the values of its derivatives at x_1, by forming only two (M) developments, one with the origin x_1 and the other with the origin x_3. If the function is zero at x, as well as all its derivatives, these (M) developments are identically zero and the function is zero at x_2. From what has been said further back it can be concluded that if a monogenic function is zero at every point of an arc, however small (at all points of this arc interior to C), when there exists on this arc at least one point interior to Γ_p a limit of points interior to Γ_p, the function being zero at all these points is zero, as well as its derivatives, at one point of Γ_p at least, and consequently identically zero in Γ_p (whatever p may be) and identically zero in C. These new monogenic functions possess then the fundamental property of analytic functions.

IV. THE LOGARITHMIC POTENTIAL

In the preceding work we have considered singular isolated points, corresponding in the physical point of view to the hypothesis of an infinite density at certain points; a statement can easily be given in which the density is everywhere finite.

Consider a regular uniform analytic function zero at infinity. If Σ is a circle such that all the singular points of the function are inside Σ, if ζ is any point outside Σ,

$$F(\zeta)=\frac{1}{2\pi i}\int_{\Sigma}\frac{F(z)dz}{\zeta-z}$$

the integration being taken in the direct sense.

Let Σ_1 and Σ_2 be two concentric circles outside Σ, let a be the center of these circles, ρ_1 and ρ_2 their radii.

Evidently, if ρ is contained between ρ_1 and ρ_2,

$$F(\zeta)=\frac{1}{2\pi}\int_0^{2\pi}\frac{F(a+\rho e^{i\alpha})\rho e^{i\alpha}d\alpha}{\zeta-a-e^{i\alpha}}.$$

If we multiply this equality by $(\rho_2-\rho)^m(\rho-\rho_1)^n$ and integrate between the limits ρ_1 and ρ_2, the expression becomes

$$F(\zeta)\int_{\rho_1}^{\rho_2}(\rho_2-\rho)^m(\rho-\rho_1)^n d\rho$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\frac{F(a+\rho e^{i\alpha})(\rho_2-\rho)^m(\rho-\rho_1)^n\rho e^{i\alpha}d\alpha d\zeta}{\zeta-a-\rho^{i\alpha}}$$

Put

$$\int_{r_1}^{r_2}(\rho_2-\rho)^m(\rho-\rho_1)^n dr=\frac{1}{2\pi}H$$

$$a+\rho e^{i\alpha}=x+iy.$$

Then

$$F(\zeta)=\frac{1}{H}\iint_{C_1C_2}\frac{F(x+iy)(\rho_2-\rho)^m(\rho-\rho_1)^n}{\zeta-x-iy}e^{i\alpha}dxdy$$

or putting

$$\zeta=\xi+i\eta$$

$$(19)\qquad F(\zeta)=\theta(\xi,\eta)=\iint_{C_1C_2}\frac{\phi(x,y)\,dxdy}{\xi+i\eta-x-iy}$$

the region of integration being the ring contained between the circles C_1 and C_2.

We shall define the function $\phi(x, y)$ outside this ring by giving it the value zero; the whole plane can then be taken as the region of integration. The function $\phi(x, y)$ is bounded and continuous in the whole plane; its derivatives are also bounded, at least as far as order m on C_1 and as far as order n on C_2; by an artifice analogous to that which we are about to employ, it would be easy to arrange matters so that all the derivatives would be continuous; in general it is enough to know that the derivatives are continuous as far as some order, fixed beforehand.

If the function $F(z)$ has a singular point a, ρ_1 can be made to tend to zero and if, further, the product $\rho^m F(z)$ remains finite for $z = a$, the formula holds for $\rho_1 = 0$; if this product does not remain finite, in the formula we replace $(\rho - \rho_1)^m$

by $e^{-\frac{1}{\rho}}$ or $e^{-e^{\frac{1}{\rho}}}$ etc. Further, in the case of a unique singular point, the circle C_2 can be drawn with a radius as small as we please, after the circle C_1 has been reduced to zero.

It is easy to deduce from this that every regular analytic uniform function, zero at infinity, can be represented in every region D interior to its region of existence W, and approaching W as nearly as we wish, by an expression of the form

$$F(\zeta)=\theta(\xi,\eta)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{\phi(x,y)dxdy}{\xi+i\eta-x-iy} \tag{20}$$

the function $\phi(x, y)$ being bounded and, further, zero at all points of D (this hypothesis involves the fact that $\phi(x, y)$ is zero at infinity, since the point at infinity belongs to D).

Inversely every expression of the form (20) in which $\phi(x, y)$ is a bounded function, zero at infinity, and continuous in the whole plane, as well as its derivatives (at least up to order m), represents a function which is monogenic at every point where $\phi(x, y)$ is zero; for by a simple calculation,

$$\frac{\partial\theta}{\partial\xi}+i\frac{\partial\theta}{\partial\eta}=2\pi\phi(\xi,\eta).$$

If the points where $\phi(x,y)$ is zero form a W region, the theory of analytic functions shows us that the function $\theta(\xi, \eta)$ is determined at every point of W by the knowledge of its values in the neighborhood of any particular point of W. The problem of the general determination of the region of existence of monogenic functions can then be set as follows: to find the conditions which $\phi(x, y)$ should satisfy in order that this fundamental property of $\theta(\xi, \eta)$ should hold; that is, that the knowledge of this function on an arc of a curve where it is monogenic allows the calculation of its value in the whole region of monogeneity.

Consider, for example, the series of rational fractions with simple poles. Denoting by C_a a circle with center a and radius ρ, and taking a point ζ outside this circle for which $|z-a|=r$, $\frac{1}{\xi-a}=\iint_{C_a}\frac{3(\rho-r)}{\pi\rho^3}\frac{dxdy}{\zeta-z}$.

When the point ζ is inside the circle C_a, the integral is easily calculated; putting $\left|\frac{\zeta-a}{\rho}\right|=\lambda$ its value is

$$\frac{3\lambda^2-2\lambda^3}{\zeta-a}.$$

The function

$$\theta(\xi,\eta;a)=\iint_{C_a}\frac{3(\rho-r)}{\pi\rho^3}\frac{dxdy}{\xi+i\eta-x-iy}$$

is then bounded in the whole plane; outside C_a it is monogenic and coincides with the analytic function $\frac{1}{\zeta-a}$.

Evidently an infinity of functions $\theta_n(\xi,\eta)$ can be defined in a similar way, such that the equation

$$\theta_n(\xi,\eta)=\frac{1}{\zeta-a_n}$$

holds for every point $\zeta=\xi+i\eta$ outside the circle C_n with center a_n and radius ρ_n, these functions being moreover *bounded* and *continuous* in the whole plane; if the $|a_n|$'s are bounded and if the coefficients A_n are such that the series

$$\Sigma\frac{|A_n|}{\rho_n^3}$$

is convergent, the series

$$\theta(\xi,\eta)=\Sigma\theta_n(\xi,\eta)$$

will be absolutely and uniformly convergent in the whole plane, and will be represented by an integral of the form

$$\theta(\xi,\eta)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{\phi(x,y)dxdy}{\xi+i\eta-x-iy}, \tag{21}$$

the function $\phi(x, y)$ being the sum of a series everywhere convergent whose respective terms are zero outside various circles C_n; this function $\phi(x, y)$ is then zero at all points exterior to all these circles and the function $\theta(\xi, \eta)$ is monogenic at these points. If the radii ρ_n are replaced by $\epsilon\rho_n$, ϵ being as small as we please, the function $\phi(x, y)$ is zero in a more and more extended region; it remains bounded, but its bound increases indefinitely as ϵ tends to zero. We are thus led to consider *a priori* a function such as (21) and to study it in the region C where $\phi(x, y)$ is zero. It is natural to suppose the region C to be simply connected; we limit ourselves to the case where this region C consists of W regions (these regions may reduce to a zero as a limiting case) and of a finite or infinite number of straight lines Δ, in such a way that any two points can be reunited by a polygonal line with a finite number of sides.

An important idea is then that of the order of infinity of the function $\phi(x, y)$ in the neighborhood of the straight lines. By a calculation analogous to that which has just been developed, the convergence of the (M) developments can be shown by making the hypothesis that $\phi(x, y)$ is not only zero upon the straight lines (which is the necessary condition of monogeneity) but tends very rapidly to zero in the neighborhood of each straight line. More precisely, if σ denotes the distance of the point (x, y) from the straight line Δ considered, it is assumed that the product

$$e^{\frac{1}{e^{\sigma^2}}}\phi(x, y)$$

tends uniformly to zero as σ tends to zero. By means of this hypothesis, it can be affirmed that the function $\theta(\xi, \eta)$ is determined in the whole region of its existence by the knowledge of its values at any point of this region. This hypothesis contains as a special case the condition satisfied by analytic functions in W regions, for if a straight line is within W, the function $\phi(x, y)$ is identically zero at all points whose distances from the straight line are less than a number σ, chosen conveniently.

The region C can be reduced to the real axis; that is the case of the function

$$\phi(\xi)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{e^{-\frac{1}{x^2}}dx\,dy}{(x^2+y^2)(\xi-x-iy)}.$$

The Taylor development

$$\theta(\xi)=\theta(\xi_0)+(\xi-\xi_0)\theta'(\xi_0)+\cdots$$

diverges for any value of ξ_0 but is summable (M), whatever ξ_0 may be, for every value of ξ, its sum being equal to the function $\theta(\xi)$. The function $\theta(\xi)$ will be called quasi-analytic.

Calculations of double integrals of form (21) lead easily to expressions of the same form; similarly in differentiating, transforming the double integral by integration by parts, it is only necessary to assume the existence of the derivatives of $\phi(x, y)$ exactly to the order of the derivatives of $\theta(\xi, \eta)$ which it is desired to calculate. To calculate the product, if we put

$$(22)\qquad \theta(\xi, \eta)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{\phi(x, y)dxdy}{\zeta-z}.$$

$$(23)\qquad \theta_1(\xi, \eta)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{\phi_1(x, y)dxdy}{\zeta-z},$$

the product becomes

$$\theta(\xi, \eta)\theta_1(\xi, \eta)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{\phi(x, y)\phi_1(x_1, y_1)dxdy\,dx_1dy_1}{(\zeta-z)(\zeta-z_1)},$$

or since

$$\frac{1}{(\zeta-z)(\zeta-z_1)}=\frac{1}{z-z_1}\left(\frac{1}{\zeta-z}-\frac{1}{\zeta-z_1}\right),$$

if we put

$$\psi(x, y)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{\phi(x_1, y_1)dx_1dy_1}{z-z_1},$$

$$\psi_1(x, y)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{\phi_1(x_1, y_1)dx_1dy_1}{z-z_1},$$

(24)

$$\theta(\xi, \eta)\theta_1(\xi, \eta)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{[\phi(x, y)\psi_1(x, y)+\psi(x, y)\phi_1(x, y)]dxdy}{\zeta-z}.$$

We can then put in the form of a double integral (24) every polynomial P in terms of one or more functions of $\theta(\xi, \eta)$ and their derivatives; if the regions of existence have a simply connected common region the differential equation obtained by equating P to zero cannot be satisfied in any portion of this region without being satisfied in the whole region C.

V. CONCLUSION

THE results we are establishing suppress the absolutely sharp demarcation established by Weierstrass's theory between real analytic functions and real non-analytic functions. I do not wish to develop the consequences of this fact from the point of view of the theory of functions; I prefer to insist a little on its importance from the point of view of the relations between mathematics and physics. It is a necessary postulate in the application of mathematics to experimental sciences, that sufficiently slight variations in the data ought not to influence the results appreciably; for, if it were not so, since the experimental data are never known with vigorous precision, one could not foresee any phenomenon. But certain mathematical properties are at least apparently discontinuous, depending for example on the fact that some number is rational or irrational. Thus the solutions of the equation

$$\frac{d^2y}{dx^2} + m^2y = \cos nx$$

are of a different nature according as the ratio $\frac{m}{n}$ is commensurable or incommensurable. Nevertheless, in this case, if $\frac{m}{n}$ varies continuously, the solution y varies very little in an interval of variation of x large compared with the length of the periods. It is not always thus, certainly, but the cases in which there is no continuity have been little studied; the equation

$$\left(\frac{dx}{dt}\right)^2 + x^2 = a^2 + 2\lambda t$$

can be given as an example in which the solutions vary discontinuously as λ becomes equal to zero; but this equation does not come under the Hamiltonian type.

It is important to know whether the properties of harmonic functions (that is, of potentials) vary continuously when the definition of the functions itself varies continuously. This has no place in Weierstrass's theory; the introduction of quasi-analytic functions restores continuity; a distribution of attracting masses infinitely near to Ox leads, if the density is *sufficiently slight* in the neighborhood of Ox, to properties of the potential on Ox which are not dissimilar from the case where the density is zero in the neighborhood of Ox.

EMILE BOREL.

THE GENERALIZATION OF ANALYTIC FUNCTIONS

ON THE THEORY OF WAVES AND GREEN'S METHOD *

THE GENERALIZATION OF ANALYTIC FUNCTIONS†

INTRODUCTION

THE generalization which is treated in the following pages has already been the subject of several investigations of mine, in the first place in several notes, published in the "Rendiconti" of the Reale Accademia dei Lincei, then in an extended memoir which appeared in the "Acta Matematica." Several of the lectures which I read at Stockholm were also devoted to this subject. And it is now my purpose, in returning to it, to consider the general case in some detail, beginning with the first foundations. In treating the general case it is necessary to consider certain elements, which I have called functions of hyperspaces, and which represent extensions of the functions of curves that I have already treated several times, in particular, in a recent course at the Sorbonne.

A space of n dimensions contains spaces of $0, 1, 2, \cdots n-1$ dimensions, and for that reason we consider functions of

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*Three lectures presented at the inauguration of the Rice Institute, by Senator Vito Volterra, Professor of Mathematical Physics and Celestial Mechanics in the University of Rome.

†Translated from the Italian by Professor Griffith Conrad Evans, of the Rice Institute.

these spaces. We shall begin by extending to these functions the fundamental concepts of continuity and differentiation, and we shall consider the condition that a function be of the first degree. This condition depends upon an extension of Stokes's theorem. We shall then consider a relation between these functions analogous to that of monogeneity, which for functions in the ordinary sense was established by Cauchy. This leads to new types of equations with functional derivatives, which present analogies with the equation of Laplace.

We can separate the functions with which we are dealing into elementary and otherwise. The former have interesting properties and applications. A certain operation of composition turns out to possess quite curious arithmetical properties.

We shall finally develop the operations of differentiation and integration, and the extension of Cauchy's theorem in complete generality.

THE GENERALIZATION OF ANALYTIC FUNCTIONS

First Lecture

GENERAL OBSERVATIONS ON HYPERSPACES — GENERAL FORMULÆ FOR MATRICES, AND RELATIONS BETWEEN THE DIRECTION COSINES OF HYPERSPACES — FUNCTIONS OF HYPERSPACES AND THEIR DERIVATIVES — EXTENSION OF STOKES'S THEOREM — CONDITIONS WHICH THE DERIVATIVES OF FUNCTIONS OF HYPERSPACES MUST SATISFY, AND FORMULÆ FOR THE TRANSFORMATION OF COÖRDINATES — ISOGENEITY — CONDITIONS FOR ISOGENEITY.

1. *General observations on hyperspaces*

1. A hyperspace (space of n dimensions) will be characterized by the multiplicity of values of n independent variables $x_1, x_2, \cdots x_n$. A hyperspace S_r of r dimensions $(r<n)$, contained in it, will correspond to the multiplicity of values which the $x_1, x_2; \cdots x_n$ assume when they are constrained

by $n - r$ independent relations, or in other words, when they depend on r independent variables $\omega_1, \omega_2, \cdots \omega_r$ to which they are bound by the n relations

$$(1) \qquad \begin{cases} x_1 = x_1(\omega_1,\ \omega_2,\ \cdots\ \omega_n) \\ x_2 = x_2(\omega_1,\ \omega_2,\ \cdots\ \omega_r) \\ \\ x_n = x_n(\omega_1,\ \omega_2,\ \cdots\ \omega_r) \end{cases}$$

We assume the differentiability of the preceding relations, and obtain

$$(2) \qquad dx_i = \sum_{1}^{r}{}_s \frac{\partial x_i}{\partial \omega_s} d\omega_s \qquad (i = 1,\ 2,\ \cdots\ n).$$

2. Let us consider the matrix

$$(3) \qquad \left\| \begin{matrix} \frac{\partial x_1}{\partial \omega_1} & \frac{\partial x_2}{\partial \omega_1} & \cdots & \frac{\partial x_n}{\partial \omega_1} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \frac{\partial x_1}{\partial \omega_r} & \frac{\partial x_2}{\partial \omega_r} & \cdots & \frac{\partial x_n}{\partial \omega_r} \end{matrix} \right\|$$

Let Δ^2 be the square of this matrix, and let us assume that if the sign of Δ is given at one point, it is fixed by continuity at all other points. When the sign of Δ is given we shall say that the *direction* of the hyperspace S_r is fixed. The quantity

$$dS_r = \Delta d\omega_1 d\omega_2 \cdots d\omega_r$$

will be called the *element of the hyperspace.*

Let us take a minor determinant of the matrix (3)

$$\Delta_{i_1 i_2 \cdots i_r} = \left| \begin{matrix} \frac{\partial x_{i_1}}{\partial \omega_1} & \frac{\partial x_{i_2}}{\partial \omega_1} & \cdots & \frac{\partial x_{i_r}}{\partial \omega_1} \\ \cdot & \cdot & \cdot & \cdot \\ \frac{\partial x_{i_1}}{\partial \omega_r} & \frac{\partial x_{i_2}}{\partial \omega_r} & \cdots & \frac{\partial x_{i_r}}{\partial \omega_r} \end{matrix} \right|$$

and write

$$\alpha_{i_1 i_2 \cdots i_r} = \frac{\Delta_{i_1 i_2 \cdots i_r}}{\Delta}. \tag{4}$$

The $\alpha_{i_1 i_2 \cdots i_r}$ will not change if we substitute for the ω_1, $\omega_2, \cdots \omega_r$ other variables bound by arbitrary relations to the first, and their signs will change only if we change the sign of the hyperspace; we shall call them the *direction cosines* of the hyperspace. We see at once that they must satisfy the relation

$$\Sigma_i \alpha^2_{i_1 i_2 \cdots i_r} = 1, \tag{A}$$

in which Σ_i denotes summation extended over all the combinations of the indices i_1, i_2, i_n.

3. If a space S_{n-r} has a point in common with S_r, and the direction cosines of S_{n-r} are denoted by $\beta_{h_1 \cdots h_{n-r}}$, we shall say that the two hyperspaces are normal to each other when we have the relation

$$\alpha_{i_1 i_2 \cdots i_r} = \beta_{h_1 h_2 \cdots h_{n-r}},$$

where all the i's are different from the h's, and the series of numbers $i_1 i_2, \cdots i_r$, h_1, $h_2, \cdots h_{n-r}$ is a permutation of the numbers 1, 2, $\cdots n$, which is always odd or always even.

4. Whatever l may be, we can write

$$d\omega_l = \Sigma_i A_{i_1 i_2 \cdots i_{r-1}} \frac{d(x_{i_1}, x_{i_2} \cdots x_{i_{r-1}})}{d(\omega_1, \omega_2 \cdots \omega_{l-1} \omega_{l+1} \cdots \omega_r)} \quad (l = 1, 2, \cdots r) \tag{5}$$

in which the sum is extended over all the combinations of the indices $i_1, i_2 \cdots i_{r-1}$, and the A's are certain, in part indeterminate, infinitesimal parameters. In fact if we form the matrix of the coefficients of the A's, among its minors will be found the $r-1$th powers of the minors of the matrix (3), and so not all the minors of that matrix can be zero. If we substitute the values (5) in the equations (2) we obtain

$$dx_s = -\Sigma_i \frac{d(x_s, x_{i_1} \cdots x_{i_{r-1}})}{d(\omega_1, \omega_2 \cdots \omega_r)} A_{i_1 i_2 \cdots i_{r-1}}. \tag{6}$$

Hence if $a_{i_1, i_2 \cdots i_{r-1}} = -\Delta A_{i_1 i_2 \cdots i_{r-1}}$ we shall have

$$dx_s = \Sigma_i a_{i_1 i_2 \cdots i_{r-1}} \alpha_{s i_1 i_2 \cdots i_{r-1}}. \tag{7}$$

5. Besides the equations (A) the α satisfy other relations, which we shall find in the next section.

2. *General formulæ about matrices. Relations between the direction cosines of a hyperspace*

1. We shall establish in this section several fundamental formulæ regarding the minors of matrices, which we shall often have occasion to use. Let us consider the two matrices

$$(1) \quad \left\| \begin{matrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{r1} & a_{r2} & \cdots & a_{rn} \end{matrix} \right\| \qquad (2) \quad \left\| \begin{matrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{p1} & a_{p2} & \cdots & a_{pn} \end{matrix} \right\|$$

the first with r rows, and the second with p rows, $(n > r \geq p)$, both however with the same elements. Let us write

$$\begin{vmatrix} a_{1i_1} & a_{1i_2} & \cdots & a_{1i_r} \\ \cdot & \cdot & \cdot & \cdot \\ a_{ri_1} & a_{ri_2} & \cdots & a_{ri_r} \end{vmatrix} = A_{i_1 i_2 \cdots i_r}, \qquad \begin{vmatrix} a_{1h_1} & a_{1h_2} & \cdots & a_{1h_p} \\ \cdot & \cdot & \cdot & \cdot \\ a_{ph_1} & a_{ph_2} & \cdots & a_{ph_p} \end{vmatrix} = B_{h_1 h_2 \cdots h_p}$$

and consider

$$\Delta_s = \begin{vmatrix} a_{si_1} & a_{si_2} & \cdots & a_{si_{r+1}} \\ a_{1i_1} & a_{1i_2} & \cdots & a_{1i_{r+1}} \\ \cdot & \cdot & \cdot & \cdot \\ a_{ri_1} & a_{ri_2} & \cdots & a_{ri_{r+1}} \end{vmatrix} = 0 \qquad (s = 1, 2, \cdots p).$$

We shall have

$$
\begin{aligned}
0 &= \sum_{1}^{p}{}_{s} \frac{\partial B_{h_1 h_2 \cdots h_p}}{\partial a_{s h_1}} \Delta_s \\
&= -\sum_{1}^{p} \frac{\partial B_{h_1 h_2 \cdots h_p}}{\partial a_{s h_1}} \sum_{1}^{r+1}{}_{i} (-1)^{i} a_{s i_i} A_{i_1 \cdot \cdot i_{i-1} i_{i+1} \cdots i_{r+1}} \\
&= -\sum_{1}^{r+1}{}_{i} A_{i_1 \cdots i_{i-1} i_{i+1} \cdots i_{r+1}} \sum_{1}^{p} (-1)^{s} a_{s i_i} \frac{\partial B_{h_1 \cdots h^{p}}}{\partial a_{s h_1}}.
\end{aligned}
$$

From this it follows that

$$\sum_{1}^{r+1}{}_{i} (-1)^{i} A_{i_1 \cdots i_{i-1} i_{i+1} \cdots i_{r+1}} B_{i_i h_2 \cdots h_p} = 0. \tag{3}$$

2. This is the formula which we wished to obtain. In particular, if we take as identical the two matrices (1) and (2), we shall have

$$\sum_{1}^{r+1}{}_{i} (-1)^{i} A_{i_1 \cdots i_{i-1} i_{i+1} \cdots i_{r+1}} A_{i_i h_2 \cdots h_r} = 0.^{*} \tag{3'}$$

Among these equations let us notice specially the following, from which the others all follow:

$$
\begin{aligned}
0 = A_{i_1 i_2 h_1 \cdots h_{r-2}} A_{i_3 i_4 h_1 \cdots h_{r-2}} &+ A_{i_4 i_2 h_1 \cdots h_{r-2}} A_{i_1 i_3 h_1 \cdots h_{r-2}} \\
&+ A_{i_1 i_4 h_1 \cdots h_{r-2}} A_{i_4 i_2 h_1 \cdot \cdot h_{r-2}}.^{\dagger}
\end{aligned}
\tag{4}
$$

3. From the preceding formulæ we see that the direction cosines of a hyperspace must satisfy the relations

$$\sum_{1}^{r+1}{}_{s} (-1)^{s} \alpha_{i_1 i_2 \cdots i_{s-1} i_{s+1} \cdots i_{r+1}} \alpha_{i_s h_2 \cdots h_r} = 0. \tag{B}$$

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* Vedi Antonelli: "Nota sulle relazioni indipendenti" ecc. ("Ann. d. Scuola Normale sup. di Pisa," Vol. III, page 71 e seg.)

† *Ibid.*, page 73.

3. *Functions of hyperspaces and their derivatives*[*]

1. A variable ϕ will be said to be a *function of the hyperspace* S_r (of r dimensions) or a function of *order* r, if to every possible hyperspace with fixed direction corresponds a value of ϕ. This correspondence will be denoted by means of the symbol $\phi = \phi \mid [S_r] \mid$. We shall assume that we are dealing only with closed hyperspaces S_r.[†]

Let us take a point P of S_r and through it draw a hyperspace S_{n-r} normal to S_r, taking in S_{n-r} a small neighborhood s of P. If we make P describe all the points of S_r we shall generate a portion of n-dimensional space, which we shall call a neighborhood of S_r. While P is describing S_r any other point P' of s describes a new hyperspace S'_r, which we shall say belongs to the neighborhood of S_r. The function $\phi \mid [S_r] \mid$ will be said to be continuous if, when we take a quantity σ arbitrarily small, we can find a neighborhood of S_r such that

$$\text{mod}\,[\phi \mid [s_r'] \mid - \phi \mid [s_r] \mid] < \sigma,$$

where S'_r belongs to that neighborhood.

Besides the continuity of $\phi \mid [S_r] \mid$ let us admit also the following property. Let us pass from the hyperspace S_r to the hyperspace S'_r by giving to each point of S_r a displacement ϵ which varies continuously from point to point. The displacement ϵ generates a hyperspace S_{r+1} of $r + 1$ dimensions, of amplitude say, σ. We shall assume that we can make $\{\phi \mid [S_r] \mid - \phi \mid [S_r] \mid\}$ less than a number chosen arbitrarily small, provided σ be less than some value σ_0.

2. With this understood, take in S_r a neighborhood s of a point P, and give to s a displacement δx_i parallel to x_i.

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[*] Vedi la mia Nota I: "Sulle funzioni dipendenti da linee." ("Atti d. R. Acc d. Lincei," Vol. III, fasc. 9.)

[†] Vedi: Betti: "Sopra gli spazii di un numero qualunque di dimensioni." (Annali di Mat., T. IV.)

Let us denote by $\delta\phi$ the corresponding variation of ϕ, and let us suppose that the value

$$\lim_{\substack{s=0\\ \delta x_i=0}} \frac{\partial\phi}{s\cdot\partial x_i} = \phi'_{x_i} \qquad (i = 1, 2, \cdots n)$$

exists. We shall call this *the derivative of ϕ with respect to x_i at the point P.* With the assumption that the ratio which appears in the left-hand member approaches its limit uniformly, with respect to all possible points P and hyperspaces S_r, and that this limit is continuous, we can easily verify the fact that if we give to every point of S_r a displacement of components δx_1, $\delta x_2, \cdots S x_n$, the corresponding variation of ϕ is given, except for infinitesimals of higher order, by the formula

(1)
$$\delta\phi = \int_{S_r} \sum_1^n{}_i \phi'_{x_i} \delta x_i dS_r.$$

3. Let us find out now what conditions the ϕ'_{x_i} must satisfy. If the displacements are such as to carry the space S_r into itself, the quantity $\delta\phi$ must vanish. Hence we must have $\delta\phi = 0$ if we take (see § 1, form 7)

$$\delta x_i = \sum_h a_{h_1h_2\cdots h_{r-1}} \alpha_{ih_1\cdots h_{r-1}}$$

whatever the quantities a may be. Hence

$$0 = \int_{S_r} \sum_h \alpha_{h_1h_2\cdots h_{r-1}} \sum_1^n{}_i \phi'_{x_i} \alpha_{ih_1\cdots h_{r-1}} dS_r$$

and from this we have

(2)
$$\sum_1^n{}_i \phi'_{x_i} \alpha_{ih_1\cdots h_{r-1}} = 0$$

for every possible combination of the indices $h_2 \cdots h_{r-1}$.

4. Since now the α satisfy the relations § 2, (B), we have

$$\sum_1^{r+1}{}_t (-1)^t \alpha_{q_th_1\cdots h_{r-1}} \alpha_{q_1\cdots q_{t-1}q_{t+1}\cdots q_{r+1}} = 0.$$

If we multiply this by an undetermined parameter $\lambda_{q_1 q_2 \cdots q_{r+1}}$ which satisfies the condition that it changes sign for every transposition of the indices, we shall have

$$
\begin{aligned}
0 &= \sum\nolimits_q \lambda_{q_1 q_2 \cdots q_{r+1}} \sum_{t=1}^{r+1} (-1)^t \alpha_{q_t h_1 \cdots h_{r-1}} \alpha_{q_1 \cdots q_{t-1} q_{t+1} \cdots q_{r+1}} \\
&= \sum_{i=1}^{n} \sum\nolimits_q \lambda_{i q_1 \cdots q_r} \alpha_{q_1 \cdots q_r} \alpha_{i h_1 \cdots h_{r-1}}
\end{aligned}
$$

and subtracting this from equation (2),

$$
0 = \sum_{i=1}^{n} \left\{ \phi'_{x_i} - \sum\nolimits_q \lambda_{i q_1 \cdots q_r} \alpha_{q_1 q_2 \cdots q_r} \right\} \alpha_{i h_1 \cdots h_{r-1}}
$$

whence

$$
\phi'_{x_i} = \sum\nolimits_q \lambda_{i q_1 \cdots q_r} \alpha_{q_1 q_2 \cdots q_r} \tag{3}
$$
[*]

From this it follows that

$$
\begin{aligned}
\delta\phi &= \int_{S_r} \sum_{i=1}^{n} \sum\nolimits_q \lambda_{i q_1 \cdots q_r} \alpha_{q_1 q_2 \cdots q_r} \delta x_i \, dS_r \\
&= \int_{S_r} \sum\nolimits_q \lambda_{q_1 q_2 \cdots q_{r+1}} \left\{ \sum_{t=1}^{r+1} (-1)^{t-1} \alpha_{q_1 q_2 \cdots q_{t-1} q_{t+1} \cdots q_{r+1}} \delta x_{q_t} \right\} dS_r.
\end{aligned}
$$

Consider now the elements dS_r and suppose drawn through every point of it a segment of components $\delta x_1 \cdots \delta x_n$. The locus of these segments will be a space S_{r+1} of $r+1$ dimensions. If the equations of the hyperspace S_r are

$$
x_i = x_i(\omega_1, \omega_2, \cdots \omega_r) \qquad (i = 1, 2, \cdots n)
$$

the equations of the hyperspace S_{r+1} will be

$$
x_i = x_i(\omega_1, \omega_2, \cdots \omega_r) + \omega_{r+1} \delta x_i \qquad (i = 1, 2, \cdots n).
$$

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[*] See my Note II: "Sulle funzioni dipendenti da linee." ("Atti della R. Acc. dei Lincei," Vol. III, fasc. 10.)

Let us form the matrix

$$(4)\qquad \left\Vert \begin{matrix} \frac{\partial x_1}{\partial \omega_1}, & \frac{\partial x_2}{\partial \omega_1} & \cdots & \frac{\partial x_n}{\partial \omega_1} \\ \cdot & \cdot & \cdot & \cdot \\ \frac{\partial x_1}{\partial \omega_r}, & \frac{\partial x_2}{\partial \omega_r} & \cdots & \frac{\partial x_n}{\partial \omega_r} \\ \partial x_1, & \partial x_2 & \cdots & \partial x_n \end{matrix} \right\Vert$$

Let us denote its square by Δ_{r+1}^2, and the square of the matrix obtained from it by taking away the last line by Δ_r^2. We shall have

$$\Delta_{r+1}^2 = \Delta_r^2 \left\{ \sum_q \sum_{1}^{r+1}{}_t (-1)^{t-1} \alpha_{q_1 \cdots q_{t-1} q_{t+1} \cdots q_{r+1}} \delta x_{q_t} \right\}^2$$

We can fix the direction of S_{r+1} with respect to S_r in such a way that

$$\Delta_{r+1} = (-1)^r \Delta_r \sqrt{\sum_q \sum_{1}^{r+1}{}_t (-1)^{t-1} \alpha_{q_1 \cdots q_{t-1} q_{t+1} \cdots q_{r+1}} \delta x_{q_t}}$$

where the sign of the radical is taken as positive. If now we denote the direction cosines of S_{r+1} by $\beta_{q_1 q_2 \cdots q_{r+1}}$, which are calculated from the matrix (4), we shall have finally

$$\delta\phi = \int_{S_{r+1}} \sum_q \lambda_{q_1 q_2 \cdots q_{r+1}} \beta_{q_1 q_2 \cdots q_{r+1}} dS_{r+1}.$$

Hence if S_r is a movable hyperspace which passes from S_r' to S_r'', thus generating a S_{r+1}, we shall have

$$(5)\qquad \phi \,|\, [S_r''] \,| - \phi \,|\, [S_r'] \,| = \int_{S_{r+1}} \sum_q \lambda_{q_1 q_2 \cdots q_{r+1}} \beta_{q_1 q_2 \cdots q_{r+1}} dS_{r+1}.$$

It is well to note explicitly that besides varying from point to point of the total hyperspace (of n dimensions), the parameters λ may also vary for one and the same point according to the hyperspace to which they refer, and even for the same hyperspace one set of λ's may be substituted for another provided the relations (3) are always satisfied.

5. A function $\phi\,|[S_r]\,|$ will be said to be *regular* (or *simple*) when the following condition is satisfied. Let S_r' and S_r'' be two hyperspaces having a common portion s, whose direction is different according as it is considered as belonging to the first or the second hyperspace. Denote by S_r''' the hyperspace which we get by taking away s from the combination of S_r' and S_r''' and fix as its direction the direction of those two hyperspaces. We impose the condition

$$\phi|[S_r''']| = \phi|[S_r']| + \phi|[S_r'']|.$$

When ϕ is regular it follows immediately that if S_r decreases indefinitely in amplitude

$$\text{(C)} \qquad \lim \phi|[S_r]| = 0.$$

We have then immediately the further property that if S_r and S_r' are two hyperspaces with a common point P, whose elements at P are contained in a single S_{r+1}, of $r+1$ dimensions,

$$\text{(6)} \qquad \sum\nolimits_q (\lambda_{q_1q_2\cdots q_{r+1}} - \lambda'_{q_1q_2\cdot\cdot q_{r+1}})\,\beta_{q_1q_2\cdots q_{r+1}} = 0$$

where λ and λ′ are the parameters which correspond to $\phi|[S_r]|$ and $\phi|[S_r']|$ at the point P, and the β's are the direction cosines of S_{r+1}.

Upon this basis let us consider a hyperspace S_r passing through the point P, whose element at P is defined by the equations

$$dx_i = \sum_{1}^{r}{}_s\, a_{is}\,d\omega_s \qquad (i = 1, 2, \cdots n)$$

and let $S_r^{(i_1\cdots i_r)(h_1\cdots h_p)}$ denote hyperspaces passing through P defined by the equations

$$dx_{i_s} = a_{i_s,s}\,d\omega_s + \sum_{1}^{p}{}_t\, a_{i_s,h_t}\,d\omega_{h_t} \qquad \begin{cases} s = 1, 2, \cdots r \\ s \neq h_1, h_2 \cdots h_p. \end{cases}$$

$$dx_{i_v} = \sum_{1}^{p}{}_t\, a_{i_v,h_t}\,d\omega_{h_t} \qquad (v = h_1, h_2 \cdots h_p, r+1, \cdots n.)$$

In particular let us consider the hyperspaces $S^{(i_1 \cdots i_{s-1} i_{s+1} \cdots i_{r+1})}$ and $S_r^{(i_1 \cdots i_{t-1} i_{t+1} \cdots i_{r+1})}$ whose elements at P are contained in a hyperspace of $r+1$ dimensions, of which the direction cosines β are zero, except $\beta_{i_1 i_2 \cdots i_{r+1}} = 1$. By means of (6), we have

$$\lambda^{(i_1 \cdots i_{s-1} i_{s+1} \cdots i_{r+1})}_{i_1 i_2 \cdots i_{r+1}} = \lambda^{(i_1 \cdots i_{t-1} i_{t+1} \cdots i_{r+1})}_{i_1 i_2 i \cdots i_{r+1}}$$

where the indices $i_1 i_2 \cdots i_r$ denote the parameters λ corresponding to the hyperspace $S_r^{(i_1 \cdots i_r)}$. Therefore we can suppress the indices and write simply

$$\lambda^{(i_1 \cdots i_{s-1} i_{s+1} \cdots i_{r+1})}_{i_1 i_2 \cdots i_{r+1}} = \Lambda_{i_1 i_2 \cdots i_{r+1}}. \tag{7}$$

6. Two hyperspaces $S_r^{(i_1 \cdots i_r)(h_1 \cdots h_{p-1})}$ and $S_r^{(i_1 \cdots i_r)(h_1 \cdots h_p)}$ have elements at P which are contained in a S_{r+1}, whose element at P is defined by the equations

$$dx_{i_s} = a_{i_s, s}\, d\omega_s + \sum_1^p{}_t\, a_{i_s, h_t}\, d\omega_{h_t} \qquad \begin{cases} s = 1, 2, \cdots r \\ s \neq h_1, h_2, \cdots h_p \end{cases}$$

$$dx_{i_{h_p}} = a_{i_{h_p}, h_p}\, d\omega_{r+1} + \sum_1^p{}_t\, a_{i_{h_p}, h_t}\, d\omega_{h_t}$$

$$dx_{i_v} = \sum_1^p{}_t\, a_{i_v, h_t}\, d\omega_{h_t}, \qquad (v = h_1, h_2 \cdots h^{p-1}, r+1, \cdots n.)$$

Hence, if we denote by β the direction cosines of S_{r+1} and by $\alpha^{(i_1 \cdots i_r)(h_1 \cdots h_p)}$ the direction cosines of $S_r^{(i_1 \cdots i_r)(h_1 \cdots h_p)}$, we shall have

$$\frac{\beta_{i_{h_p} m_1 m_2 \cdots m_r}}{\alpha^{(i_1 \cdots i_r)(h_1 \cdots h_p)}_{m_1 m_2 \cdots m_r}} = \kappa,$$

where κ is independent of the indices $m_1, m_2, \cdots m_r$, and all the β's are zero, in the indices of which i_{h_p} is missing. From this it follows by reason of (6) that

$$\sum{}_m \left(\lambda^{(i_1 \cdots i_r)(h_1 \cdots h_p)}_{i_{h_p} m_1 \cdots m_r} - \lambda^{(i_1 \cdots i_r)(h_1 \cdots h_{p-1})}_{i_{h_p} m_1 \cdots m_r} \right) \alpha^{(i_1 \cdots i_r)(h_1 \cdots h_p)}_{m_1 \cdots m_r} = 0,$$

where the index $(i_1 \cdots i_r)(h_1 \cdots h_p)$, affixed to the λ, means that refers to the hyperspace having the same index. We have then

$$\sum_m \lambda_{i_{h_p} m_1 \cdots m_r}^{(i_1 \cdots i_r)(h_1 \cdots h_p)} \alpha_{m_1 \cdots m_r}^{(i_1 \cdots i_r)(h_1 \cdots h_p)} = \sum_m \lambda_{i_{h_p} m_1 \cdots m_r}^{(i_1 \cdots i_r)(h_1 \cdots h_{p-1})} \alpha_{m_1 \cdots m_r}^{(i_1 \cdots i_r)(h_1 \cdots hp)},$$

in which, by means of (3), we can substitute for the $\lambda_{i_{h_p} m_1 \cdots m_r}^{(i_1 \cdots i_r)(h_1 \cdots hp)}$ the $\lambda_{i_{h_p} m_1 \cdots m_r}^{(i_1 \cdots i_r)(h_1 \cdots h_{p-1})}$, and consequently, the $\Lambda_{i_{h_p} m_1 \cdots m_r}$ of formula (7).

We observe however that the hyperspace $S_r^{(i_1 \cdots i_r)(i_1 \cdots i_p)}$ is nothing but the hyperspace S_r, and therefore we can take for the λ's belonging to this space, at the point P, the λ's without index of formula (6). We have then the theorem

If ϕ is a regular function of the hyperspace S_r, contained in a hyperspace S_n, there exist for every point of S_n a system of values which can be considered as the parameters $\lambda_{i_1 i_2 \cdots i_{r+1}}$ for all the hyperspaces S_r which pass through that point.

7. From the equations (5) (C), assuming that $\phi \,|\, [S_r] \,|$ is regular we get,

$$(5') \qquad \phi \,|\, [S_r] \,| = \int_{S_{r\ 1}} \sum_q \Lambda_{q_1 q_2 \cdots q_{r+1}} \beta_{q_1 q_2 \cdots q_{r+1}} dS_{r+1}.$$

Here S_{r+1} is an arbitrary hyperspace of $r+1$ dimensions, whose boundary is S_r. If S_{r+1} grows indefinitely smaller about a point P, by writing

$$S_{r+1} = \int_{S_{r+1}} dS_{r+1}$$

we shall have

$$\lim \frac{\phi \,|\, [S_r] \,|}{S_{r+1}} = \sum_q \Lambda_{q_1 q_2 \cdots q_{r+1}} \beta_{q_1 q_2 \cdots q_{r+1}} = \frac{d\phi}{dS_{r+1}}$$

where the β are the direction cosines of S_{r+1} at P.

Let us take $S_{r+1} = S_{r+1}^{(i_1 i_2 \cdots i_{r+1})}$ such that at P all the direction cosines β shall be zero except $\beta_{i_1 i_2 \cdots i_{r+1}} = 1$. We shall have

$$\lim \frac{\phi\,|[S_r]|}{S_{r+1}^{(i_1 i_2 \cdots i_{r+1})}} = \Lambda_{i_1 i_2 \cdots i_{r+1}}.$$

Therefore we shall write

$$\Lambda_{i_1 i_2 \cdots i_{r+1}} = \frac{\partial \phi}{\partial (x_{i_1} x_{i_2} \cdots x_{i_{r+1}})},$$

and define this quantity as the *derivative of* ϕ *with respect to* $x_{i_1} x_{i_2} \cdots x_{i_{r+1}}$. What relations must these derivatives satisfy? Before proceeding to the search for these relations, it will be necessary to give an extension of Stokes's theorem, a subject which is dealt with in the next section.

4. *Extension of Stokes's theorem*

1. Let $L_{i_1 i_2 \cdots i_r}$ be functions of the points of the hyperspace S_n, such that every transposition of the indices creates a change of sign, and form the expression

$$(1) \qquad M_{i_1 i_2 \cdots i_{r+1}} = \sum_{1}^{r+1}{}_s (-1)^{s-1} \frac{\partial L_{i_1 \cdots i_{s-1} i_{s+1} \cdots i_{r+1}}}{\partial x_{i_s}}.$$

Let S_{r+1} be a hyperspace, of $r+1$ dimensions, bounded by a set of hyperspaces S_r, let its direction cosines be $\alpha_{i_1 i_2 \cdots i_{r+1}}$, and form the expression

$$\int_{S_{r+1}} \Omega \, dS_{r+1},$$

putting

$$\Omega = \Sigma_i M_{i_1 i_2 \cdots i_{r+1}} \alpha_{i_1 i_2 \cdots i_{r+1}}.$$

If the equations of S_{r+1} are

$$x_i = x_i(\omega_1, \omega_2 \cdots \omega_{r+1}) \qquad (i = 1, 2, \cdots n),$$

we shall have

$$\Omega dS_{r+1}$$

$$= \sum_\iota M_{\iota_1\iota_2\cdots\iota_{r+1}} \frac{d(x_{\iota_1}x_{\iota_2} \cdots x_{\iota_{r+1}})}{d(\omega_1\omega_2 \cdots \omega_{r+1})} d\omega_1 d\omega_2 \cdots d\omega_{r+1}$$

$$= \sum_\iota \sum_{1}^{n}{}_s \frac{\partial L_{\iota_1\cdots\iota_{s-1}\iota_{s+1}\cdots\iota_{r+1}}}{\partial x_{\iota_s}} \frac{d(x_{\iota_s}x_{\iota_1} \cdots x_{\iota_{s-1}}x_{\iota_{s+1}} \cdots x_{\iota_{r+1}})}{d(\omega_1\omega_2 \cdots \omega_{r+1})} d\omega_1 d\omega_2 \cdots d\omega_{r+1}.$$

$$= \sum_\iota \frac{d(L_{\iota_1\cdots\iota_r}, x_{\iota_1}x_{\iota_2} \cdots x_{\iota_r})}{d(\omega_1, \omega_2, \cdots \omega_{r+1})} d\omega_1 d\omega_2 \cdots d\omega_{r+1}$$

$$= \sum_\iota \sum_{1}^{r+1}{}_t (-1)^{t-1} \frac{\partial L_{\iota_1\cdots\iota_r}}{\partial \omega_t} \frac{d(x_1 \cdots x_{\iota_r})}{d(\omega_1 \cdots \omega_{t-1}\omega_{t+1} \cdots \omega_{r-1}} d\omega_1 d\omega_2 \cdots d\omega_{r+1}.$$

Hence

$$\int_{S_{r+1}} \Omega dS_{r+1}$$

$$= \int_{S_r} \sum_\iota \sum_{1}^{r+1}{}_t L_{\iota_1\iota_2\cdots\iota_r} \frac{d(x_{\iota_1}x_{\iota_2} \cdots x_{\iota_r})}{d(\omega_\iota \cdots \omega_{t-1}\omega_{t+1} \cdots \omega_{r+1})} d\omega_\iota \cdots d\omega_{t-1} d\omega_{t+1} \cdots d\omega_{r+1}.$$

We can make the hyperspace S depend on r independent parameters $\bar{\omega}_1, \bar{\omega}_2 \cdots \bar{\omega}_r$, whence we shall have

$$\int_{S_{r+1}} \Omega dS_{r+1} = \int_{S_r} \sum_\iota L_{\iota_1\iota_2\cdots\iota_r} \frac{d(x_{\iota_1}x_{\iota_2} \cdots x_{\iota_r})}{d(\bar{\omega}_1\bar{\omega}_2 \cdots \bar{\omega}_r)} d\bar{\omega}_1 d\bar{\omega}_2 \cdots d\bar{\omega}_r.$$

From this comes the formula

$$(2) \quad \int_{S_{r+1}} \sum_\iota M_{\iota_1\iota_2\cdots\iota_{r+1}} \alpha_{\iota_1\iota_2\cdots\iota_{r+1}} dS_{r+1} = \int_{S_r} \sum_\iota L_{\iota_1\iota_2\cdots\iota_r} \beta_{\iota_1\iota_2\cdots\iota_r} dS_r,$$

where the β's are the direction cosines of the hyperspace S_r.

2. From these formulæ it follows that if

$$\int_{S_r} \sum_\iota L_{\iota_1\iota_2\cdots\iota_r} \beta_{\iota_1\iota_2\cdots\iota_r} dS = 0,$$

for every closed hyperspace S_r in the region S_n, the necessary and sufficient conditions that must be satisfied are

$$(3) \qquad M_{i_1 i_2 \cdots i_{r+1}} = \sum_{1}^{r+1}{}_s (-1)^{s-1} \frac{\partial L_{i_1 \cdots i_{s-1} i_{s+1} \cdots i_{r+1}}}{\partial x_{i_s}} = 0$$

for every combination of the indices $i_1 i_2 \cdots i_{r+1}$.

5. *Conditions which the derivatives of functions of hyperspaces must satisfy. Formulæ for the change of coördinates*

1. Let $\phi \,|\, [S_r] \,|$ be regular, and return to formula (5′) of section 3. Since the integral which appears in the right-hand member does not change when S_{r+1} changes, provided the boundary S_r does not change, we must have

$$\int_{S_{r+1}} \sum_q \Lambda_{q_1 q_2 \cdots q_{r+1}} \beta_{q_1 q_2 \cdots q_{r+1}} \, dS_{r+1} = 0$$

when the integration is extended over any closed hyperspace S_{r+1}. Hence the necessary and sufficient conditions which the Λ must satisfy in order to be the derivatives of a regular function of hyperspaces S_r (see section 4, article 2) is

$$(\mathrm{D}) \qquad \sum_{1}^{r+2} (-1)^{s-1} \frac{\partial \Lambda_{i_1 \cdots i_{s-1} i_{s+1} \cdots i_{r+2}}}{\partial x_{i_s}} = 0$$

for every possible combination of the indices $i_1 i_2 \cdots i_{r+2}$. We can write these equations, making use of the symbols of section 3, article 7, in the form

$$(\mathrm{D}') \qquad \sum_{1}^{r+2}{}_s (-1)^{s-1} \frac{\partial}{\partial x_{i_s}} \frac{\partial \phi}{\partial (x_{i_1} \cdots x_{i_{s-1}} x_{i_{s+1}} \cdots x_{i_{r+2}})} = 0.$$

We shall call these conditions the *conditions of integrability*.

2. Consider now the formulæ for change of variable, transforming the variables $x_1, x_2 \cdots x_n$ into $x'_1, x'_2 \cdots x'_n$ by

means of the relations

$$x'_i = x'_i(x_1, x_2, \cdots x_n) \qquad (i = 1, 2, \cdots n)$$

such that

$$\frac{d(x'_1, x'_2 \cdots x'_n)}{d(x_1, x_2 \cdots x_n)}$$

is always finite and different from zero. Let us consider two regions which correspond in a one-to-one manner, S_n and S'_n, one belonging to the first set of variables, the other to the second. Let S_{r+1} be a hyperspace, bounded by S_r and contained in S_n, and let S'_{r+1}, bounded by S'_r, correspond to it in S'_n. If we suppose that S_{r+1} is given by the equations

$$x_i = x_i(\omega_1 \omega_2 \cdots \omega_{r+1}) \qquad (i = 1, 2 \cdots n),$$

we shall have

$$\phi \,|\, [S_r] \,|$$

$$= \int_{S_{r+1}} \sum_i \frac{\partial \phi}{\partial(x_{i_1} x_{i_2} \cdots x_{i_{r+1}})} \frac{d(x_{i_1} \cdots x_{i_{r+1}})}{d(\omega_1 \cdots \omega_{r+1})} d\omega_1 d\omega_2 \cdots d\omega_{r+1}$$

$$= \int_{S_{r+1}} \sum_i \frac{\partial \phi}{\partial(x_{i_1} \cdots x_{i_{r+1}})} \sum_h \frac{d(x_{i_1} \cdots x_{i_{r+1}})}{d(x'_{h_1} \cdots x'_{h_{r+1}})} \frac{d(x'_{h_1} \cdots x'_{h_{r+1}})}{d(\omega_1 \cdots \omega_{r+1})} d\omega_1 \cdots d\omega_{r+1}$$

$$= \int_{S_{r+1}} \sum_h \frac{d(x'_{h_1} \cdots x'_{h_{r+1}})}{d(\omega_1 \cdots \omega_{r+1})} \sum_i \frac{\partial \phi}{\partial(x_{i_1} \cdots x_{i_{r+1}})} \frac{d(x_{i_1} \cdots x_{i_{r+1}})}{d(x'_{h_1} \cdots x'_{h_{r+1}})} d\omega_1 \cdots d\omega_{r+1}$$

$$= \int_{S'_{r+1}} \sum_h \beta'_{h_1 \cdots h_{r+1}} \left(\sum_i \frac{\partial \phi}{\partial(x_{i_1} \cdots x_{i_{r+1}})} \frac{d(x_{i_1} \cdots x_{i_{r+1}})}{d(x'_{h_1} \cdots x'_{h_{r+1}})} \right) dS'_{r+1}$$

where the β' denote the direction cosines of S'_{r+1}.

If we write

$$\Lambda'_{h_1 h_2 \cdots h_{r+1}} = \sum_i \frac{\partial \phi}{\partial(x_{i_1} \cdots x_{i_{r+1}})} \frac{d(x_{i_1} \cdots x_{i_{r+1}})}{d(x'_{h_1} \cdots x'_{h_{r+1}})}$$

we shall have

$$\phi \,|\, [S_r] \,| = \phi \,|\, [S'_r] \,| = \int_{S'_{r+1}} \sum_h \Lambda'_{h_1 h_2 \cdots h_{r+1}} \beta'_{h_1 h_2 \cdots h_{r+1}} dS'_{r+1},$$

whence

$$\Lambda'_{h_1 \cdots h_{r+1}} = \frac{\partial \phi}{\partial(x'_{h_1} x'_{h_2} \cdots x'_{h_{r+1}})}.$$

The desired formulæ for the transformation of coördinates become then

$$(1)\qquad \frac{\partial\phi}{\partial(x'_{h_1}x'_{h_2}\cdots x'_{h_{r+1}})}=\sum\nolimits_{i}\frac{\partial\phi}{\partial(x_{i_1}x_{i_2}\cdots x_{i_{r+1}})}\frac{d(x_{i_1}x_{i_2}\cdots x_{i_{r+1}})}{d(x'_{h_1}x'_{h_2}\cdots x'_{h_{r+1}})}.$$

3. If we multiply the preceding equations by

$$\frac{d(x_{s_{r+2}}x_{s_{r+3}}\cdots x_{s_n})}{d(x'_{h_{r+2}}x_{h_{r+3}}\cdots x_{h_n})}$$

and add them, for all values of the h's, we shall have

$$\sum\nolimits_{h}\frac{\partial\phi}{\partial(x'_{h_1}x'_{h_2}\cdots x'_{h_{r+1}})}\frac{d(x_{s_{r+2}}x_{s_{r+3}}\cdots x_{s_n})}{d(x'_{h_{r+2}}x'_{h_{r+3}}\cdots x_{h_n})}$$

$$=\frac{\partial\phi}{\partial(x_{s_1}x_{s_2}\cdots x_{s_{r+1}})}\frac{d(x_1x_2\cdots x_n)}{d(x'_1x'_2\cdots x'_n)}$$

where

$$(h_1, h_2\cdots h_{r+1}, h_{r+2}\cdots h_n)\equiv(s_1, s_2\cdots s_{r+1}, s_{r+2}\cdots s_n)\equiv(1, 2, \cdots n),$$

the notation being used to denote the fact that the groups of the h's and of the s's are two even permutations of the first m integers. Hence

$$(2)\qquad \frac{\partial\phi}{\partial(x_{s_1}x_{s_2}\cdots x_{s_{r+1}})}$$

$$=\frac{1}{\dfrac{d(x_1x_2\cdots x_n)}{d(x'_1x'_2\cdots x'_n)}}\sum\nolimits_{h}\frac{\partial\phi}{\partial(x'_{h_1}x'_{h_2}\cdots x'_{h_{r+1}})}\frac{d(x_{s_{r+2}}x_{s_{r+3}}\cdots x_{s_n})}{d(x'_{h_{r+2}}x_{h_{r+3}}\cdots x'_{h_n})}.$$

4. By means of the equations (D′), which are satisfied by the functions $\dfrac{\partial\phi}{\partial(x_{s_1}\cdots x_{s_{r+1}})}$, and the analogous equations satisfied by the functions $\dfrac{\partial\phi}{\partial(x'_{h_1}\cdots x'_{h_{r+1}})}$, we obtain the theorem:

If the quantities $a_{i_1 i_2 \cdots i_{r+1}}$ *(which change sign for every transposition in the indices) satisfy the equations*

$$(3) \qquad \sum_{1}^{r+2}{}_{s} (-1)^{s-1} \frac{\partial a_{i_1 \cdots i_{s-1} i_{s+1} \cdots i_{r+2}}}{\partial x_{i_s}} = 0$$

the quantities $a'_{h_1 h_2 \cdots h_{r+1}}$ *given by the formulæ*

$$a'_{h_1 h_2 \cdots h_{r+1}} = \frac{1}{\dfrac{d(x'_1 \cdots x'_n)}{d(x_1 \cdots x_n)}} \sum{}_{i} a_{i_1 i_2 \cdots i_{r+1}} \frac{d(x'_{h_{r+2}} x'_{h_{r+3}} \cdots x'_{h_n})}{d(x_{i_{r+2}} x_{i_{r+3}} \cdots x_{i_n})}$$

$$(i_1, i_2 \cdots i_n) \equiv (h_1, h_2 \cdots h_n) \equiv (1, 2, \cdots n)$$

will satisfy the analogous equations

$$(3') \qquad \sum_{1}^{r+2}{}_{s} (-1)^{s-1} \frac{\partial a'_{h_1 \cdots h_{s-1} h_{s+1} \cdots h_{r+2}}}{\partial x'_{h_s}} = 0.$$

5. Let us write $\dfrac{\partial \phi}{\partial (x_{i_1} \cdots x_{i_{r+1}})} = a_{i_1 \cdots i_{r+1}}.$

We wish to show that if the following conditions are satisfied

$$(4) \qquad \sum_{1}^{r+2} (-1)^{s} a_{i_1 \cdots i_{s-1} i_{s+1} \cdots i_{r+2}} a_{i_s h_1 \cdots h_r} = 0$$

and we make a change of variables from the $x_1, x_2 \cdots x_n$ to the $x'_1, x'_2, \cdots x'_n$, we shall obtain the result that the quantities

$$a'_{h_1 \cdots h_{r+1}} = \frac{\partial \phi}{\partial (x'_{h_1} \cdots x'_{h_{r+1}})}$$

will satisfy the analogous equations

$$(4') \qquad \sum_{1}^{r+2}{}_{s} (-1)^{s} a'_{h_1 \cdots h_{s-1} h_{s+1} \cdots h_{r+2}} a'_{h_s t_1 \cdots t_r} = 0.$$

In fact if we have the relations (4), the $a_{i_1 \cdots i_{r+1}}$ will be minor determinants of a matrix

$$\left\|\begin{matrix} A_{1,1} & A_{1,2} \cdots A_{1,n} \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\ A_{r+1,1} & A_{r+1,2} \cdots A_{r+1,n} \end{matrix}\right\|,$$

that is, we can write

$$a_{i_1 \cdots i_{r+1}} = \begin{vmatrix} A_{1,i_1} & \cdots & A_{1,i_{r+1}} \\ \cdot & \cdot & \cdot \\ A_{r+1,i_1} & \cdots & A_{r+1,i_{r+1}} \end{vmatrix}.$$

If we write $$\frac{\partial x_s}{\partial x'_i} = B_{is}$$

we shall have, by means of (1), the equations

$$a'_{h_1 \cdots h_{r+1}} = \sum_i \begin{vmatrix} A_{1,i_1} & \cdots & A_{1,i_{r+1}} \\ \cdot & \cdot & \cdot \\ A_{r+1,i_1} & \cdots & A_{r+1,i_{r+1}} \end{vmatrix} \begin{vmatrix} B_{h_1 i_1} & \cdots & B_{h_1, i_{r+1}} \\ \cdot & \cdot & \cdot \\ B_{h_{r+1}, i_1} & \cdots & B_{h_{r+1}, i_{r+1}} \end{vmatrix}$$

that is, if we define $\sum_1^\infty {}_s A_{is} B_{hs} = C_{ih}$, the relations

$$a'_{h_1 \cdots h_{r+1}} = \begin{vmatrix} C_{1,h_1} & \cdots & C_{1,h_{r+1}} \\ C_{r+1,h_1} & \cdots & C_{r+1,h_{r+1}} \end{vmatrix}.$$

In other words, the quantities $a'_{h_1 \cdots h_{r+1}}$ are minor determinants of the matrix

$$\left\|\begin{matrix} C_{11} & C_{12} \cdots C_{1n} \\ C_{21} & C_{22} \cdots C_{2n} \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\ C_{r+1,1} & C_{r+1,2} \cdots C_{r+1,n} \end{matrix}\right\|$$

and so the equations (4′) will be satisfied.

When the equations (4) are satisfied, the function $\phi|[S_r]|$ is said to be *elementary* (see §§ 10, 14).

6. *Isogeneity*[*]

1. Two complex functions f, ϕ, of hyperspaces S_r, which are *regular*, are said to be *isogenous* if in every point of the total hyperspace S_n, the ratio

$$\frac{\dfrac{d\phi}{dS_{r+1}}}{\dfrac{df}{dS_{r+1}}}$$

is independent of the hyperspace S.

Separating the real and imaginary parts, let us write

$$\frac{\partial f}{\partial(x_{i_1}x_{i_2}\cdots x_{i_{r+1}})} = p_{i_1\cdots i_{r+1}} + iq_{i_1\cdots i_{r+1}} = p_I + iq_I$$

$$\frac{\partial \phi}{\partial(x_{i_1}x_{i_2}\cdots x_{i_{r+1}})} = \omega_{i_1\cdots i_{r+1}} + i\chi_{i_1\cdots i_{r+1}} = \overline{\omega}_I + i\chi_I$$

where I denotes the set of indices $i_1 i_2 \cdots i_{r+1}$, that is, $I \equiv (i_1 \cdots i_{r+1})$. The necessary and sufficient condition in order that f and ϕ be isogenous may be written

$$(1) \qquad \frac{\overline{\omega}_I + \chi_I}{p_I + q_I} = \frac{\overline{\omega}_H + i\chi_H}{p_H + iq_H},$$

where $H \equiv (h_1 h_2 \cdots h_{r+1})$ is another arbitrary combination of the indices. From the preceding equations we find

$$(2) \qquad \begin{cases} \overline{\omega}_I p_H - \overline{\omega}_H p_I = \chi_I q_H - \chi_H q_I, \\ \overline{\omega}_I q_H - \overline{\omega}_H q_I = \chi_H p_I - \chi_I p_H. \end{cases}$$

2. Let us write
$$p_I p_H + q_I q_H = E_{I,H},$$
$$p_I q_H - p_H q_I = D_{I,H}.$$

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[*] See my note: "*Sopra una estensione della teoria di Riemann sulle funzioni di variabile complessa.*" ("Atti della R. Acc. dei Lincei," Vol. III, fasc. 10.)

Among the E's and D's we shall have the relations

$$D_{IH}E_{LK}+D_{HK}E_{LI}+D_{KI}E_{LH}=0, \tag{3}$$

$$\begin{vmatrix} E_{IH}E_{IL} \\ E_{KH}E_{KL} \end{vmatrix} = \begin{vmatrix} p_Ip_H+q_Iq_H & p_Ip_L+q_Iq_L \\ p_Kp_H+q_Kq_H & p_Kp_L+q_Kq_L \end{vmatrix} = \begin{vmatrix} p_Iq_I \\ p_Kq_K \end{vmatrix} \begin{vmatrix} p_Hq_H \\ p_Lq_L \end{vmatrix} = D_{IK}D_{HL}$$

whence

$$E_{IH}E_{KL}-E_{KH}E_{IL}=D_{IK}D_{HL}. \tag{4}$$

3. If we solve the equations (2) for $\overline{\omega}_I$ and χ_I, we shall have

$$\overline{\omega}_I=\frac{E_{IH}\chi_I-E_{II}\chi_H}{D_{HI}}, \qquad \chi_I=\frac{E_{IH}\overline{\omega}_I-E_{II}\overline{\omega}_H}{D_{IH}}$$

Since, however, the first member of these equations does not depend on H, we must have

$$\begin{aligned}\overline{\omega}_I&=\frac{E_{IH}\chi_I-E_{II}\chi_H}{D_{HI}}=\frac{E_{IK}\chi_I-E_{II}\chi_K}{D_{KI}}\\&=\frac{(E_{IH}\chi_I-E_{II}\chi_H)E_{IK}-(E_{IK}\chi_I-E_{II}\chi_K)E_{IH}}{D_{HI}E_{IK}-D_{KI}E_{IH}}\\&=\frac{E_{IH}\chi_K-E_{IK}\chi_H}{D_{HK}}.\end{aligned}$$

In a similar way we can operate on the expression for χ_I, and therefore whatever H and K may be we have the formulæ

$$\overline{\omega}_I=\frac{E_{IH}\chi_K-E_{IK}\chi_H}{D_{HK}}, \quad \chi_I=\frac{E_{IH}\overline{\omega}_K-E_{IK}\overline{\omega}_H}{D_{KH}}. \tag{E}$$

4. From the preceding formulæ it follows that

$$\begin{aligned}D_{HK}\overline{\omega}_I&=E_{IH}\chi_K-E_{IK}\chi_H,\\D_{KI}\overline{\omega}_H&=E_{HK}\chi_I-E_{HI}\chi_K,\\D_{IH}\overline{\omega}_K&=E_{KI}\chi_H-E_{KH}\chi_I,\end{aligned}$$

hence, whatever I, H, K may be, we have the formula

$$\text{(F)} \qquad D_{HK}\bar{\omega}_I + D_{KI}\bar{\omega}_H + D_{IH}\bar{\omega}_K = 0,$$

and similarly, $D_{HK}\chi_I + D_{KI}\chi_H + D_{IH}\chi_K = 0.$

5. Let us return to the equations (E); from them it follows that

$$(5) \quad \Theta_{IL} = \frac{1}{D_{IL}}\begin{vmatrix} \bar{\omega}_I & \chi_I \\ \bar{\omega}_L & \chi_L \end{vmatrix} = \frac{\chi_K E_{IH} - \chi_H E_{IK}}{D_{IL}D_{HK}}\chi_L - \frac{\chi_K E_{LH} - \chi_H E_{LK}}{D_{IL}D_{HK}}\chi_I$$

$$= \frac{E_{IH}\chi_K\chi_L - E_{IK}\chi_H\chi_L + E_{LK}\chi_H\chi_I - E_{LH}\chi_K\chi_I}{D_{IL}D_{HK}}.$$

If we interchange I with H and L with K the last member of this equation will not change. Hence we shall have

$$(6) \qquad \frac{1}{D_{IL}}\begin{vmatrix} \bar{\omega}_I & \chi_I \\ \bar{\omega}_L & \chi_L \end{vmatrix} = \frac{1}{D_{HK}}\begin{vmatrix} \bar{\omega}_H & \chi_H \\ \bar{\omega}_K & \chi_K \end{vmatrix}.$$

In other words, the quantities Θ_{IL} are independent of I and L, and so we can denote them all by Θ.

If in (5) we put $I = H$, $L = K$, we shall have

$$(7) \qquad \Theta = \frac{E_{II}\chi_L^2 - 2\,E_{IL}\chi_I\chi_L + E_{LL}\chi_I^2}{D_{IL}^2}$$

$$= \frac{(p_I\chi_L - p_L\chi_I)^2 + (q_I\chi_L - q_L\chi_I)^2}{D_{IL}^2}$$

formulæ which show that Θ is a *positive* quantity. If in (5) we interchange $\bar{\omega}$ and χ, and p and q, the Θ will not change, and we shall have for Θ the alternative expression

$$(5') \qquad \Theta = \frac{E_{IH}\bar{\omega}_K\bar{\omega}_L - E_{IK}\bar{\omega}_H\bar{\omega}_L + E_{LK}\bar{\omega}_H\bar{\omega}_I - E_{LH}\omega_K\omega_I}{D_{IL}D_{HK}}.$$

If we write $\phi = \phi_1 + i\phi_2$ and make use of our symbols $I, H \cdots$, we can write

$$\omega_I = \overline{\omega}_{i_1 \cdots i_{r+1}} = \frac{\partial \phi_1}{\partial(x_{i_1} x_{i_2} \cdots x_{i_{r+1}})} = \frac{\partial \phi_1}{\partial(x_I)},$$

$$\chi_1 = \chi_{i_1 \cdots i_{r+1}} = \frac{\partial \phi_2}{\partial(x_{i_1} x_{i_2} \cdots x_{i_{r+1}})} = \frac{\partial \phi_2}{\partial(x_I)}$$

where (x_I) is a substitute for $(x_{i_1} x_{i_2} \cdots x_{i_{r+1}})$, *i.e.*

$$(x_I) \equiv (x_{i_1} x_{i_2} \cdots x_{i_{r+1}}).$$

The expression for Θ can now be written

(G) $\Theta =$

$$\frac{E_{IH}\dfrac{\partial\psi}{\partial(x_I)}\dfrac{\partial\psi}{\partial(x_L)} - E_{IK}\dfrac{\partial\psi}{\partial(x_H)}\dfrac{\partial\psi}{\partial(x_K)} + E_{LK}\dfrac{\partial\psi}{\partial(x_H)}\dfrac{\partial\psi}{\partial(x_I)} - E_{LH}\dfrac{\partial\psi}{\partial(x_K)}\dfrac{\partial\psi}{\partial(x_I)}}{D_{IL}D_{HK}}$$

where in place of ψ we can put either ϕ_1 or ϕ_2.

6. We know that the quantities $\overline{\omega}$ and χ must satisfy the following equations (see section 5, article 1)

$$\sum_{1}^{r+2}{}_s (-1)^{s-1} \frac{\partial}{\partial x_s} \overline{\omega}_{i_1 \cdots i_{s-1} i_{s+1} \cdots i_{r+2}} = 0, \quad \sum_{1}^{r+2}{}_s (-1)^{s-1} \frac{\partial}{\partial x_s} \chi_{i_1 \cdots i_{s-1} i_{s+1} \cdots i_{r+2}},$$

and therefore, from (E), we have the following equations

$$(H) \left\{ \begin{aligned} &\sum_{1}^{r+2}{}_s (-1)^{s-1} \frac{\partial}{\partial x_s} \\ &\qquad \left\{ \frac{\chi_K E_{i_1 \cdots i_{s-1} i_{s+1} \cdots i_{r+2},\ H} - \chi_H E_{i_1 \cdots i_{s-1} i_{s+1} \cdots i_{r+2},\ K}}{D_{HK}} \right\} = 0 \\ &\sum_{1}^{r+2}{}_s (-1)^{s-1} \frac{\partial}{\partial x_s} \\ &\qquad \left\{ \frac{\overline{\omega}_K E_{i \cdots i_{s-1} i_{s+1} \cdots i_{r+2},\ H} - \overline{\omega}_H E_{i_1 \cdots i_{s-1} i_{s+1} \cdots i_{r+2},\ K}}{D_{HK}} \right\} = 0 \end{aligned} \right.$$

or, by reason of (H) and (F), ϕ_1 and ϕ_2 must satisfy the equations

$$(H')\quad \sum_{1}^{r+2}{}_{\iota_s}(-1)^{s-1}\frac{\partial}{\partial x_s}\left[\frac{E_{\iota_1\cdots\iota_{s-1}\iota_{s+1}\cdots\iota_{r+2},\ H}\dfrac{\partial\psi}{\partial(x_K)}-E_{\iota_1\cdots\iota_{s-1}\iota_{s+1}\cdots\iota_{r+2},\ K}\dfrac{\partial\psi}{\partial(x_H)}}{D_{HK}}\right]=0$$

$$(F')\qquad D_{HK}\frac{\partial\psi}{\partial(x_I)}+D_{KI}\frac{\partial\psi}{\partial(x_H)}+D_{IH}\frac{\partial\psi}{\partial(x_K)}=0.$$

7. Conversely it can be shown that if $\psi|[S_r]|$ is a real regular function and satisfies the preceding equations, it may be considered as the real part of a function $\psi+i\theta$ isogenous to f. In fact, by means of (H') we can write

$$\frac{E_{I,H}\dfrac{\partial\psi}{d(x_K)}-E_{I,K}\dfrac{\partial\psi}{\partial(x_H)}}{D_{HK}}=\frac{\partial\theta_{H,K}}{\partial(x_I)}$$

where $(x_I)=(x_{\iota_1}\cdots x_{\iota_{s-1}}x_{\iota_{s+1}}\cdots x_{\iota_{r+2}})$. But from (F') and (3) it follows that the first member of the preceding equations is independent of H and K, hence we can take the θ_{HK} as independent of their subscripts and write them all equal to θ, so that

$$\frac{E_{IH}\dfrac{\partial\psi}{\partial(x_K)}-E_{IK}\dfrac{\partial\psi}{\partial(x_H)}}{D_{HK}}=\frac{\partial\theta}{\partial(x_I)}.$$

And now if from these equations we follow the inverse procedure to that of articles 1, 2, 3, we find that the ratio

$$\frac{\dfrac{\partial(\psi+i\theta)}{\partial(x_I)}}{p_I+iq_I}$$

is independent of the indices (I), so that $\psi+i\theta$ is isogenous to f. The equations (H') and (F') operate in our case in the same way as the equation $\Delta^2 = 0$ in the theory of Riemann.

7. *Conditions for isogeneity.*

1. If we take arbitrarily a regular function of hyperspaces S_r it will not always be possible to associate with it an isogenous function. In order for that it is necessary that certain conditions be satisfied. In fact if $F\,|\,[S_r]\,|$ is a regular function to which $\Phi\,|\,[S_r]\,|$ is isogenous, and we write

$$\frac{\partial F}{\partial(x_{i_1} \cdots x_{i_{r+1}})} = p_{i_1 \cdots i_{r+1}}, \qquad \frac{\partial \Phi}{\partial(x_{i_1} \cdots x_{i_{r+1}})} = \overline{\omega}_{i_1 \cdots i_{r+1}},$$

we must have

$$\frac{\overline{\omega}_{i_1 \cdots i_{r+1}}}{p_{i_1 \cdots i_{r+1}}} = \phi$$

where ϕ is independent of the indices $i_1 \cdots i_{r+1}$. Hence it follows that

$$\overline{\omega}_{i_1 \cdots i_{r+1}} = \phi p_{i_1 \cdots i_{r+1}}$$

so that

$$0 = \sum_1^{r+2}{}_s (-1)^s \frac{\partial \overline{\omega}_{i_1 \cdots i_{s-1} i_{s+1} \cdots i_{r+2}}}{\partial x_{i_s}} = \sum_1^{r+2}{}_s (-1)^s \frac{\partial(\phi p_{i_1 \cdots i_{s-1} i_{s+1} \cdots i_{r+2}})}{\partial x_{i_s}}$$

$$= \sum_1^{r+2}{}_s (-1)^s p_{i_1 \cdots i_{s-1} i_{s+1} \cdots i_{r+1}} \frac{\partial \phi}{\partial x_{i_s}}.$$

From this we conclude that *it is necessary and sufficient in order that there may exist a function isogenous to* $F\,|\,[S_r]\,|$ *that the system of simultaneous linear differential equations*

$$(1) \qquad \sum_1^{r+2}{}_s (-1)^s p_{i_1 \cdots i_{s-1} i_{s+1} \cdots i_{r+2}} \frac{\partial \phi}{\partial x_{i_s}} = 0$$

admit solutions.

It is for this reason that in § 9 we shall study systems of differential equations of this form. In the meantime let us observe that the equations (1) may in some cases be incompatible. Thus, if we have in four dimensions the regular function $F\,|\,[S_1]\,|$, the equations (1) become

$$-p_{23}\frac{\partial\phi}{\partial x_1}+p_{13}\frac{\partial\phi}{\partial x_2}-p_{12}\frac{\partial\phi}{\partial x_3}=0,$$

$$-p_{34}\frac{\partial\phi}{\partial x_2}+p_{24}\frac{\partial\phi}{\partial x_3}-p_{23}\frac{\partial\phi}{\partial x_4}=0,$$

$$-p_{41}\frac{\partial\phi}{\partial x_3}+p_{31}\frac{\partial\phi}{\partial x_4}-p_{34}\frac{\partial\phi}{\partial x_1}=0,$$

$$-p_{12}\frac{\partial\phi}{\partial x_4}+p_{42}\frac{\partial\phi}{\partial x_1}-p_{41}\frac{\partial\phi}{\partial x_2}=0,$$

and these equations will be incompatible unless

$$p_{12}p_{34}+p_{13}p_{42}+p_{14}p_{23}=0$$

2. We now proceed to prove the following theorem:

The necessary and sufficient condition in order that equations (1) *admit a common solution ϕ is that we can write*

$$p_{i_1\cdots i_{r+2}}=\sum_{1}^{r+1}{}_{s}(-1)^s\frac{\partial\phi}{\partial x_{i_s}}\frac{\partial\psi}{\partial(x_{i_1}\cdots x_{i_{s-1}}x_{i_{s+1}}\cdots x_{i_{r+1}})} \tag{2}$$

where ψ is a regular function of hyperspaces.

Let us write $\dfrac{\partial\psi}{\partial(x_{i_1}\cdots x_{i_r})}=q_{i_1\cdots i_r}.$

It is easy to show that if the equations

$$p_{i_1\cdots i_{r+1}}=\sum_{1}^{r+1}{}_{s}(-1)^s\frac{\partial\phi}{\partial x_{i_s}}q_{i_1\cdots i_{s-1}i_{s+1}\cdots i_{r+1}} \tag{2'}$$

are satisfied, the equations (1) will also be satisfied. In fact, we shall have

$$\sum_{1}^{r+2}{}_{t}(-1)^t p_{i_1\cdots i_{t-1}i_{t+1}\cdots i_{r+2}}\frac{\partial\phi}{\partial x_{i_t}}$$

$$=\sum_{1}^{r+2}{}_{t}\sum_{1}^{r+2}{}^{(t)}_{s}(-1)^{s'+t}\frac{\partial\phi}{\partial x_{i_s}}\frac{\partial\phi}{\partial x_{i_t}}q_{i_1\cdots i_{s-1}i_{s+1}\cdots i_{t-1}i_{t+1}\cdots i_{r+1}},$$

in which $\sum_{1\ s}^{r+2\ (t)}$ is extended over all the values of the index s from 1 to $r+2$, the value t excepted, and s' should be taken equal to s or to $s-1$ according as $s<t$ or $s>t$. Hence the left-hand member of the equation is zero, and the equations (1) are satisfied. From (2′) it also follows easily that

$$\sum_{1\ s}^{r+2}(-1)^s\frac{\partial p_{i_1\cdots i_{s-1}i_{s+1}\cdots i_{r+2}}}{\partial x_{i_s}}=0.$$

Thus we have shown that our condition is sufficient. To show that it is also necessary, let us execute a change of variables, instead of $x_1, x_2 \cdots x_n$ taking $x'_1=\phi$, $x'_2=x_2 \cdots x'_n=x_n$. If we prime the letters which refer to the new variables, we shall have

1st) if $\qquad i_1, i_2, \cdots i_r \neq 1$

$$q_{i_1\cdots i_r}=q'_{i_1\cdots i_r}+\sum_{1\ t}^{r}(-1)^{t-1}q'_{1i_1\cdots i_{t-1}i_{t+1}\cdots i_r}\frac{\partial\phi}{\partial x_{i_t}}$$

2d) if $\qquad i_h=1$

$$q_{i_1\cdots i_r}=\sum_{1\ s}^{r}(-1)^{s-1}q'_{1i_1\cdots i_{s-1}i_{s+1}\cdots i_r}\frac{\partial\phi}{\partial x_{i_s}}=(-1)^{h-1}q'_{1i_1\cdots i_{h-1}i_{h+1}\cdots i_r}\frac{\partial\phi}{\partial x_{i_h}}.$$

Supposing momentarily that $i_1 \cdots i_{r+1} \neq 1$ we shall have

$$p_{i_1\cdots i_{r+1}}=\sum_{1\ s}^{r+1}(-1)^s\frac{\partial\phi}{\partial x_{i_s}}q'_{i_1\cdots i_{s-1}i_{s+1}\cdots i_{r+1}}$$

$$+\sum_{1\ s}^{r+1}(-1)^s\frac{\partial\phi}{\partial x_{i_s}}\sum_{1\ t}^{r+1\ (s)}(-1)^{t'}q'_{1i_1\cdots i_{t-1}i_{t+1}\cdots i_{s-1}i_{s+1}\cdots i_{r+1}}\frac{\partial\phi}{\partial x_{i_t}}$$

where $t'=\begin{cases}t-1\\ t\end{cases}$ according as $\begin{cases}t<s\\ t>s.\end{cases}$

Hence

$$p_{i_1\cdots i_{r+1}}=\sum_{1\ s}^{r+1}(-1)^s q'_{i_1\cdots i_{s-1}i_{s+1}\cdots i_{r+1}}\frac{\partial\phi}{\partial x_{i_s}}. \tag{3}$$

If we suppose instead that some one of the indices of p is equal to 1, say $i_1 = 1$, we shall have

$$(3') \qquad p_{1i_2\cdots i_{r+1}} = -\frac{\partial\phi}{\partial x_1} q'_{i_2\cdots i_{r+1}} - \frac{\partial\phi}{\partial x_1}\sum_{2}^{r+1}{}_{t}(-1)^t q'_{1i_2\cdots i_{t-1}i_{t+1}\cdots i_{r+1}}\frac{\partial\phi}{\partial x_{i_t}}$$
$$+\sum_{2}^{r+1}{}_{s}(-1)^s q'_{1i_2\cdots i_{s-1}i_{s+1}\cdots i_{r+1}}\frac{\partial\phi}{\partial x_1}\frac{\partial\phi}{\partial x_{i_s}} = -\frac{\partial\phi}{\partial x_1} q'_{i_2\cdots i_{r+1}}.$$

We shall show that (3) is a consequence of (3′). In fact, from (3′) we have

$$q'_{i_2\cdots i_{r+1}} = -\frac{p_{1i_2\cdots i_{r+1}}}{\left(\frac{\partial\phi}{\partial x_1}\right)}$$

so that (3) becomes

$$p_{i_1\cdots i_{r+1}} = -\sum_{1}^{r+1}{}_{s}(-1)^s \frac{p_{1i_1\cdots i_{s-1}i_{s+1}\cdots i_{r+1}}}{\frac{\partial\phi}{\partial x_1}}\frac{\partial\phi}{\partial x_{i_s}}$$

and if we put $i_0 = 1$, this gives us

$$\sum_{0}^{r+1}{}_{s}(-1)^s p_{i_0i_1\cdots i_{s-1}i_{s+1}\cdots i_r}\frac{\partial\phi}{\partial x_{i_s}} = 0,$$

an equation which is identically true.

We must now prove that the functions

$$q'_{i_2\cdots i_{r+1}} = -\frac{p_{1i_2\cdots i_{r+1}}}{\frac{\partial\phi}{\partial x_1}}$$

satisfy the conditions of integrability (see section 5, article 1), assuming therein that ϕ is constant.

We have in fact (see section 5, article 3)

$$p'_{1i_2\cdots i_{r+1}} = \frac{1}{\frac{d(\phi x_2\cdots x_n)}{d(x_1x_2\cdots x_n)}}\Sigma_h p_{h_1h_2\cdots h_{r+1}}\frac{d(x_{i_{r+2}}\cdots x_{i_n})}{d(x_{h_{r+2}}\cdots x_{h_n})}$$

where

$$(h_1h_2\cdots h_{r+1}h_{r+2}\cdots h_n) \equiv (1\,i_2\cdots i_{r+1}i_{r+2}\cdots i_n) \equiv (1, 2, \cdots n)$$

so that
$$p'_{1i_2\cdots i_{r+1}} = \frac{p_{1i_2\cdots i_{r+1}}}{\left(\frac{\partial\phi}{\partial x_1}\right)}.$$

If $i_1, i_2 \cdots i_{r+1} \neq 1$, we have (see section 5, article 3)

$$p'_{i_1 \cdots i_{r+1}} = \frac{1}{\dfrac{d(\phi x_2 \cdots x_n)}{d(x_1 x_2 \cdots x_n)}} \Sigma_h p_{h_1 \cdots h_{r+1}} \frac{d(x_{i_{r+2}} \cdots x_{i_n})}{d(x_{h_{r+2}} \cdots x_{h_n})}$$

$$= \frac{1}{\dfrac{d\phi}{dx_1}} \sum_0^{r+1} (-1)^s p_{i_0 i_1 \cdots i_{s-1} i_{s+1} \cdots i_{r+1}} \frac{\partial \phi}{\partial x_{i_s}} = 0.$$

And so if we apply the theorem of section 5, article 3, we shall have

$$0 = \sum_2^{r+1} (-1)^s \frac{\partial}{\partial x_{i_s}} \left[\frac{p_{1, i_2 \cdots i_{s-1} i_{s+1} \cdots i_{r+1}}}{\left(\dfrac{\partial \phi}{\partial x_1} \right)} \right]$$

$$= \sum_2^{r+1} (-1)^s \frac{\partial}{\partial x_{i_s}} q'_{i_2 \cdots i_{s-1} i_{s+1} \cdots i_{r+1}}.$$

The functions q' then satisfy the conditions of integrability, and it will be possible to determine a function ψ which satisfies equations (2). Thus it is shown that the given condition is necessary.

3. Given the F for which (1) is satisfied, the ψ which satisfies (2) is not determined. We shall see how all the ψ's which satisfy (2) may be found when one of them, ψ_1, is known. If ψ_1 and ψ satisfy (2), and we write

$$\psi - \psi_1 = \psi_2, \quad \frac{\partial \psi_2}{\partial (x_{i_1} \cdots {}_{i_r})} = q^{(2)}_{i_1 \cdots i}$$

we shall have

$$0 = \sum_{1}^{r+1}{}_s (-1)^s \frac{\partial \phi}{\partial x_{i_s}} q^{(2)}_{i_1 \cdots i_r}$$

and therefore

$$q^{(2)}_{i_1 \cdots i_r} = \sum_1^r{}_s (-1)^s \frac{\partial \phi}{\partial x_{i_s}} \frac{\partial \Theta}{\partial (x_{i_1} \cdots x_{i_{s-1}} x_{i_{s+1}} \cdots x_{i_r})}.$$

in which $\Theta | [S_{r-1}] |$ is arbitrary.

THE GENERALIZATION OF ANALYTIC FUNCTIONS

Second Lecture

EXPRESSIONS FOR ISOGENOUS FUNCTIONS — AUXILIARY REMARKS ON SYSTEMS OF SIMULTANEOUS DIFFERENTIAL EQUATIONS — ON THE ELEMENTARY FUNCTIONS — COMPOSITION OF FUNCTIONS OF HYPERSPACES — NEW CONSIDERATIONS WITH REFERENCE TO THE RELATION OF ISOGENEITY — DIFFERENTIATION AND INTEGRATION — ISOGENEITY OF ORDER r.

8. *Expressions for isogenous functions*

1. If $F|[S_r]|$ and $\Phi|[S_r]|$ are isogenous, it follows from what has been shown in the preceding section that we can write

$$\frac{\partial F}{\partial(x_{i_1}\cdots x_{i_{r+1}})} = p_{i_1\cdots i_{r+1}} = \sum_{1}^{r+1}{}_s(-1)^s \frac{\partial f}{\partial x_{i_s}} \frac{\partial\psi}{\partial(x_{i_1}\cdots x_{i_{s-1}}x_{i_{s+1}}\cdots x_{i_{r+1}})}$$

$$\frac{\partial \Phi}{\partial(x_{i_1}\cdots x_{i_{r+1}})} = \overline{\omega}_{i_1\cdots i_{r+1}} = \sum_{1}^{r+1}{}_s(-1)^s \frac{\partial \phi}{\partial x_{i_s}} \frac{\partial\psi}{\partial(x_{i_1}\cdots x_{i_{s-1}}x_{i_{s+1}}\cdots x_{i_{r+1}})},$$

where $\psi|[S_{r-1}]|$ is regular and ϕ is a function of f; and we know that the ratio $\frac{\overline{\omega}_{i_1\cdots i_{r+1}}}{p_{i_1\cdots i_{r+1}}}$ (independent of the indices) is equal to $\frac{d\phi}{df}$.

2. Let us write

$$L_{i_1\cdots i_{r+1}} = f\frac{\partial\psi}{\partial(x_{i_1}\cdots x_{i_r})}.$$

It follows that

(1) $$p_{i_1\cdots i_{r+1}} = \sum_{1}^{r+1}{}_s(-1)^s \frac{\partial L_{i_1\cdots i_{s-1}i_{s+1}\cdots i_{r+1}}}{\partial x_{i_s}}.$$

If now S_{r+1} is a space of $r+1$ dimensions whose boundary is S_r, we shall have

$$F|[S_r]| = \int_{S_{r+1}} \Sigma_i p_{i_1\cdots i_{r+1}} \alpha_{i_1\cdots i_{r+1}} dS_{r+1},$$

where the $\alpha_{i_1\cdots i_{r+1}}$ are the direction cosines of S_{r+1}. And if we substitute for the p's their values (1) and apply the extension of Stokes's theorem (see Section 4), we shall have

(2) $$F|[S_r]| = \int_{S_r} f \frac{d\psi}{dS_r} dS_r$$

and similarly,

(2′) $$\Phi|[S_r]| = \int_{S_r} \phi \frac{d\psi}{dS_r} dS_r.$$

3. Conversely, if F and Φ are given by the preceding formulae, with $\phi=\phi(f)$, the F and Φ must be isogenous.

9. *Auxiliary remarks on systems of simultaneous differential equations*

1. Consider the system of differential equations

(1) $$H_{i_1 i_2\cdots i_{r+2}} = \sum_{1}^{r+2}{}_s (-1)^s A_{i_1\cdots i_{s-1} i_{s+1}\cdots i_{r+2}} \frac{\partial\phi}{\partial x_{i_s}} = 0$$

whose coefficients satisfy the conditions

(2) $$\sum_{1}^{r+2}{}_s (-1)^s A_{i_s h_1\cdots h_{r+1}} A_{i_1\cdots i_{s-1} i_{s+1}\cdots i_{r+2}} = 0$$

and are such that they cnange sign with every transposition of the indices. With this convention, if we have an H with two of its indices equal, its value must be zero.

2. Among the A's, one at least must be different from zero. If $A_{i_1 i_2 \cdots i_{r+1}}$ is such a one, *all the equations* (1) *will follow from the equations (independent among themselves).*

$$(3) \qquad H_{i_1 \cdots i_{r+1} h_1} = 0, \quad H_{i_1 \cdots i_{r+1} h_2} = 0, \; \cdots \; H_{i_1 \cdots i_{r+1} h_{n-r-1}} = 0,$$

in which none of the $h_1, h_2 \cdots h_{n-r-1}$ *is equal to another, or to an* i.

Let us take, in fact, the system

$$(4) \qquad H_{i_1 \cdots i_{r+1} k_1} = 0, \quad H_{i_1 \cdots i_{r+1} k_2} = 0, \; \cdots \; H_{i_1 \cdots i_{r+1} k_{r+2}} = 0,$$

where the k_s are arbitrary. If a k_s is equal to one of the i_l, the corresponding equation will be an identity; otherwise, it will be one of the equations (3). The equations (4) can be written in the form

$$A_{i_1 \cdots i_{r+1}} \frac{\partial \phi}{\partial x_{k_s}} + \sum_{1}^{r+1}{}_l (-1)^l A_{k_s i_1 \cdots i_{l-1} i_{l+1} \cdots i_{r+1}} \frac{\partial \phi}{\partial x_{i_l}} = 0.$$

If we multiply each one by $(-1)^s A_{k_1 \cdots k_{s-1} k_{s+1} \cdots k_{r+2}}$ and add them together for all values of the subscript s from 1 to $r+2$, we shall have

$$A_{i_1 \cdots i_{r+1}} \sum_{1}^{r+2}{}_s (-1)^s A_{k_1 k_2 \cdots k_{s-1} k_{s+1} \cdots k_{r+2}} \frac{\partial \phi}{\partial x_{k_s}}$$
$$+ \sum_{1}^{r+1}{}_l (-1)^l \frac{\partial \phi}{\partial x_{i_l}} \sum_{1}^{r+2}{}_s A_{k_s i_1 \cdots i_{l-1} i_{l+1} \cdots i_{r+2}} A_{k_1 \cdots k_{s-1} k_{s+1} \cdots k_{r+2}} = 0,$$

whence $\displaystyle\sum_{1}^{r+2} (-1)^s A_{k_1 k_2 \cdots k_{s-1} k_{s+1} \cdots k_{r+2}} \frac{\partial \phi}{\partial x_{k_s}} = H_{k_1 \cdots k_{r+2}} = 0$

so that the theorem is proved.

3. Now let us form the alternating function of Poisson

$$(H_{i_1 i_2 \cdots i_{r+2}}, H_{h_1 h_2 \cdots h_{r+2}})$$

taking $i_1 = h_1$, $i_2 = h_2$, $\cdots$ $i_{r+1} = h_{r+1}$ and writing $h_{r+2} = i_{r+3}$,

we shall have

$$(H_{h_1 \cdots h_{r+2}}, H_{i_1 \cdots i_{r+2}})$$

$$= \sum_{1}^{r+2}{}_{t} \sum_{1}^{r+2}{}_{s} (-1)^{s+t} \left(A_{h_1 \cdots h_{s-1} h_{s+1} \cdots h_{r+2}} \frac{\partial A_{i_1 \cdots i_{t-1} i_{t+1} \cdots i_{r+2}}}{\partial x_{h_s}} \frac{\partial \phi}{\partial x_{i_t}} \right)$$

$$- \sum_{1}^{r+2}{}_{s} (-1)^{s+t} \left(A_{i_1 \cdots i_{s-1} i_{s+1} \cdots i_{r+2}} \frac{\partial A_{h_1 \cdots h_{t-1} h_{t+1} \cdots h_{r+2}}}{\partial x_{i_s}} \frac{\partial \phi}{\partial x_{ht}} \right)$$

$$= \sum_{1}^{r+1}{}_{t} \sum_{1}^{r+1}{}_{s} (-1)^{s+t} \frac{\partial (A_{i_1 \cdots i_{r+1}} A_{i_1 \cdots i_{s-1} i_{s+1} \cdots i_{t-1} i_{t+1} \cdots i_{r+3}})}{\partial x_{i_s}} \frac{\partial \phi}{\partial x_{i_t}}$$

$$- \sum_{1}^{r+2}{}_{s} (-1)^{s} \frac{\partial A_{h_1 \cdots h_{s-1} h_{s+1} \cdots h_{r+2}}}{\partial x_{i_s}} \sum_{1}^{r+2}{}_{t} (-1)^{t} A_{i_1 \cdots i_{t-1} i_{t+1} \cdots i_{r+3}} \frac{\partial \phi}{\partial x_{i}}$$

$$+ \sum_{1}^{r+2}{}_{s} (-1)^{s} \frac{\partial A_{i_1 \cdots i_{s-1} i_{s+1} \cdots i_{r+2}}}{\partial x_{i_s}} \sum_{1}^{r+2}{}_{t} (-1)^{t} A_{h_1 \cdots h_{t-1} h_{t+1} \cdots h_{r+2}} \frac{\partial \phi}{\partial x_{h_t}}$$

$$+ \sum_{1}^{r+2}{}_{t} (-1)^{t+r+2} \frac{\partial (A_{i_1 \cdots i_{r+1}} A_{i_1 \cdots i_{t-1} i_{t+1} \quad r+2})}{\partial x_{i_{r+2}}} \frac{\partial \phi}{\partial x_{i_t}}$$

$$- \sum_{1}^{r+2}{}_{t} (-1)^{t+r+2} \frac{\partial (A_{i_1 \cdots i_{r+1}} A_{h_1 \cdots h_{t-1} h_{t+1} \cdots h_{r+2}})}{\partial x_{i_{r+2}}} \frac{\partial \phi}{\partial x_{h_t}}$$

$$+ \sum_{1}^{r+2}{}_{s} (-1)^{s+r+2} \frac{\partial (A_{i_1 \cdots i_{r+1}} A_{h_1 \cdots h_{s-1} h_{s+1} \cdots h_{r+2}})}{\partial x_{h_s}} \frac{\partial \phi}{d x_{i_r+2}}$$

$$- \sum_{1}^{r+2}{}_{s} (-1)^{s+r+2} \frac{\partial (A_{i_1 \cdots i_{r+1}} A_{i_1 \cdots i_{s-1} i_{s+1} \cdots i_{r+2}})}{\partial x_{i_s}} \frac{\partial \phi}{\partial x_{h_{r+2}}}.$$

If we write

$$\sum_{1}^{r+2}{}_{s} (-1)^{s} \frac{\partial A_{h_1 \cdots h_{s-1} h_{s+1} \cdots h_{r+2}}}{\partial x_{h_s}} = L_{h_1 \cdots h_{r+2}},$$

we shall have

$$(H_{h_1 \cdots h_{r+2}} H_{i_1 \cdots i_{r+2}})$$

$$= A_{i_1 \cdots i_{r+1}} \sum_{1}^{r+3}{}_{s} (-1)^{s} L_{i_1 \cdots i_{s-1} i_{s+1} \cdots i_{r+3}} \frac{\partial \phi}{\partial x_{i_t}}$$

$$+ \sum_{1}^{r+3}{}_{s} (-1)^{s} \frac{\partial A_{i_1 \cdots i_{r+1}}}{\partial x_{i}} H_{i_1 \cdots i_{s-1} i_{s+1} \cdots i_{r+3}} + L_{i_1 \cdots i_{r+2}} H_{i_1 \cdots i_{r+1} i_{r+3}}$$

$$- L_{i_1 \cdots i_{r+1} i_{r+3}} H_{i_1 \cdots i_{r+2}}.$$

Hence to the system (1) we must add the equations

$$\sum_{1}^{r+1}{}_{s}(-1)^{s} L_{i_1 \cdots i_{s-1} i_{s+1} \cdots i_{r+3}} \frac{\partial \phi}{\partial x_{i_s}} = 0$$

so that *if the conditions*

$$L_{i_1 \cdots i_{r+2}} = \sum_{1}^{r+2}{}_{s}(-1)^{s} \frac{\partial A_{i_1 \cdots i_{s-1} i_{s+1} \cdots i_{r+2}}}{\partial x_{h_s}} = 0$$

are satisfied for every combination of the indices $i_1 \cdots i_{r+2}$, the *system* (1) *will be complete.*

4. From this it follows that the equations (1) of section 7 will form a complete system whenever, in addition to the conditions of integrability (see section 5, article 1), the functions p satisfy also the following conditions:

$$\sum_{1}^{r+2}{}_{s}(-1)^{s} p_{i_1 \cdots i_{s-1} i_{s+1} \cdots i_{r+2}} p_{i_s h_1 \cdots h_r} = 0. \tag{5}$$

Hence for *elementary* functions (see section 5, article 5) the system of equations (1) of section 7 is *complete.*

10. *The elementary functions*

1. Let us suppose that the function $F \mid [S_r] \mid$ is regular and elementary, so that the system (1) of section 7, or the equivalent system (3) of section 9, is complete. There will exist then $r + 1$ independent integrals

$$\phi, \phi_1, \cdots \phi_r.$$

Hence the ratio

$$\theta = \frac{p_{i_1 \cdots i_r}}{\left(\dfrac{d(\phi, \phi_1, \cdots \phi_r)}{d(x_{i_1} \cdots x_{i_{r+1}})}\right)} = \frac{\left(\dfrac{dF}{d(x_{i_1} \cdots x_{i_{r+1}})}\right)}{\left(\dfrac{d(\phi, \phi_1 \cdots \phi_r)}{d(x_{i_1} \cdots x_{i_{r+1}})}\right)}$$

will be independent of the subscripts $i_1 \cdots i_r$, and we shall have

$$p_{i_1 \cdots i_{r+1}} = \theta \frac{d(\phi, \phi_1 \cdots \phi_r)}{d(x_{i_1} \cdots x_{i_{r+1}})}.$$

But we must have $\sum_1^{r+2}{}_s(-1)^s\dfrac{\partial p_{i_1\cdots i_{s-1}i_{s+1}\cdots i_{r+2}}}{\partial x_{i_s}}=0,$

so that $$\sum_1^{r+2}{}_s(-1)^s\frac{\partial\theta}{\partial x_{i_s}}\frac{d(\phi,\phi_1\cdots\phi_r)}{d(x_{i_1}\cdots x_{i_{s-1}}x_{i_{s+1}}\cdots x_{i_{r+2}})}=0,$$

and consequently $$\sum_1^{r+2}{}_s(-1)^s p_{i_1\cdots i_{s-1}i_{s+1}\cdots i_{r+2}}\frac{\partial\theta}{\partial x_{i_s}}=0.$$

The quantity θ will therefore be a function of $\phi_0, \phi_1, \cdots \phi_r$, and if we write $\dfrac{\partial\phi_0}{\partial\phi}=\theta$, we shall have

$$p_{i_1\cdots i_{r+1}}=\frac{\partial\phi_0}{\partial\phi}\frac{d(\phi,\phi_1\cdots\phi_r)}{d(x_{i_1}\cdots x_{i_{r+1}})}=\frac{d(\phi_0,\phi_1\cdots\phi_r)}{d(x_{i_1}\cdots x_{i_{r+1}})}.$$

We have therefore the following theorem:

If F is an elementary function, it follows that

$$\frac{\partial F}{\partial(x_{i_1}\cdots x_{i_{r+1}})}=\frac{d(\phi_0,\phi_1\cdots\phi_r)}{d(x_{i_1}\cdots x_{i_{r+1}})}=p_{i_1\cdots i_{r+1}}$$

where $\phi_0, \phi_1, \cdots \phi_r$ are independent integrals of the complete system

$$(1)\qquad \sum_1^{r+2}{}_s(-1)^s p_{i_1\cdots i_{s-1}i_{s+1}\cdots i_{r+2}}\frac{\partial\phi}{\partial x_{i_s}}=0.$$

2. Conversely, *if we take $r+1$ functions $\phi_0, \phi_1, \cdots \phi_r$ and write*

$$\frac{d(\phi_0,\phi_1,\cdots\phi_r)}{d(x_{i_1}\cdots x_{i_{r+1}})}=p_{i_1\cdots i_{r+1}},$$

the quantities $p_{i_1\cdots i_{r+1}}$ will be the derivatives of an elementary function. In fact, they will satisfy the conditions of integrability, and also the conditions (5) of the preceding section (see section 5, article 5).

We shall say that the functions $\phi_0, \phi_1, \cdots \phi_r$ are *conjugate* to the function F, and that F is conjugate to them.

3. If Φ is isogenous to F, and we write

$$\frac{\partial \Phi}{\partial(x_{\iota_1} \cdots x_{\iota_{r+1}})} = \overline{\omega}_{\iota_1 \cdots \iota_{r+1}},$$

we must have

$$\frac{\overline{\omega}_{\iota_1 \cdots \iota_{r+1}}}{p_{\iota_1 \cdots \iota_{r+1}}} = \psi,$$

ψ being an integral of equation (1). Hence ψ must be a function of $\phi_0, \phi_1, \cdots \phi_r$. If we take $\psi = \dfrac{\partial \lambda}{\partial \phi}$, we shall have

$$\overline{\omega}_{\iota_1 \cdots \iota_{r+1}} = \frac{\partial \lambda}{\partial \phi} \frac{d(\phi, \phi_1, \cdots \phi_r)}{d(x_{\iota_1} \cdots x_{\iota_{r+1}})} = \frac{d(\lambda, \phi_1, \cdots \phi_r)}{d(x_{\iota_1} \cdots x_{\iota_{r+1}})},$$

from which we deduce the theorem:

All the functions isogenous to an elementary function are themselves elementary.

4. If we apply to the elementary functions the formula (2), section 8, relative to the possibility of defining isogenous functions, we have

$$F\,|\,[S_r]\,| = \int_{S_r} \phi \frac{d(\phi_0, \phi_1 \cdots \phi_r)}{d(\omega_1 \cdots \omega_r)}\, d\omega_1 \cdots d\omega_r, \tag{2}$$

where

$$x_1 = x_1(\omega_1, \cdots \omega_r),\ x_2 = x_2(\omega_1, \cdots \omega_r), \cdots x_n = x_n(\omega_1, \cdots \omega_r),$$

the equations of the hyperspace S_r.

11. *The composition of functions of hyperspaces*

1. The results which we have obtained in the preceding section can be expressed in a different form by means of special symbols. That is what we shall do in this section, after having proved a fundamental theorem.

Let $F\,|\,[S_r]\,|$ and $\Phi\,|\,[S_{\iota-r}]\,|$ be two regular functions of hyperspaces, and write

$$\frac{dF}{d(x_{h_1} \cdots x_{h_{r+1}})} = p_{h_1 \cdots h_{r+1}}, \quad \frac{\partial \phi}{\partial(x_{h_{r+2}} \cdots x_{h_{t+2}})} = q_{h_{r+2} \cdots h_{t+2}}.$$

$$(\text{I}) \qquad m_{i_1 \cdots i_{t+2}} = \sum\nolimits_h (-\text{I})^{\binom{h_1 \cdots h_{t+2}}{i_1 \cdots i_{t+2}}} p_{h_1 \cdots h_{r+1}} q_{h_{r+2} \cdots h_{t+2}},$$

in which $h_1 \cdots h_{t+2}$ is a permutation of $i_1 \cdots i_{t+2}$; the sum Σ_h is extended over all the combinations of the $t+2$ subscripts $i_1 \cdots i_{t+2}$, $r+\text{I}$ at a time; and the symbol $(-\text{I})^{\binom{h_1 \cdots h_{t+2}}{i_1 \cdots i_{t+2}}}$ represents $+\text{I}$ or $-\text{I}$, according as the substitution which appears in the exponent is even or odd.

2. We shall show that *there exists a regular function* $\Psi|[S_{t+1}]|$, *such that*

$$\frac{\partial \Psi}{\partial(x_{i_1} \cdots x_{i_{t+2}})} = m_{i_1 \cdots i_{t+2}}.$$

In fact, the quantities m satisfy the conditions of integrability (section 5, article I); that is,

$$\sum_{1}^{t+3} (-\text{I})^s \frac{\partial m_{i_1 \cdots i_{s-1} i_{s+1} \cdots i_{t+3}}}{\partial x_{i_s}}$$

$$= \sum_{1}^{t+3}{}_s (-\text{I})^s \sum\nolimits_h (-\text{I})^{\binom{h_1 \cdots h_{t+2}}{i_1 \cdots i_{s-1} i_{s+1} \cdots i_{t+3}}} \frac{\partial}{\partial x_{i_s}} (p_{h_1 \cdots h_{r+1}} q_{h_{r+2} \cdots h_{t+3}}) = 0.$$

3. To represent the fact that the relation (I) holds among the three functions F, Φ, Ψ we shall write

$$\Psi \equiv (F, \Phi).$$

We have immediately

$$(F, \Phi) \equiv (-\text{I})^{(r+1)(t-r+1)} (\Phi, F).$$

If $\Theta \,|\, [S_{v-t}] \,|$ is a regular function, and we write

$$\frac{\partial \Theta}{\partial (x_{h_{t+3}} \cdots x_{h_{v+3}})} = n_{h_{t+3} \cdots h_{v+3}}$$

$$l_{i_1 \cdots i_{v+3}} = \sum\nolimits_h (-1)^{\binom{h_1 \cdots h_{v+3}}{i_1 \cdots i_{v+3}}} p_{h_1 \cdots h_{r+1}} q_{h_{r+2} \cdots h_{t+2}} n_{h_{t+3} \cdots h_{v+3}}$$

$$= \sum\nolimits_h (-1)^{\binom{h_1 \cdots h_{v+3}}{i_1 \cdots i_{v+3}}} m_{h_1 \cdots h_{t+2}} n_{h_{t+3} \cdots h_{v+3}},$$

it follows that there exists a function $\Lambda \,|\, [S_{v+2}] \,|$ which is regular, and such that

$$\frac{\partial \Lambda}{\partial (x_{i_1} \cdots x_{i_{v+3}})} = l_{i_1 \cdots i_{v+3}}.$$

We shall write $\Lambda = (F, \Phi, \Theta)$.

And in general if the functions $F^{(i)} \,|\, [S_{r_i}] \,|$ are regular, we shall understand by

(2) $$M \equiv (F^{(1)}, F^{(2)}, \cdots F^{(k)})$$

a regular function of hyperspaces S_R, $R = \sum_1^k r_i + k$, obtained as follows:

$$\Phi_2 \equiv (F^{(1)}, F^{(2)}),\ \Phi_3 \equiv (\Phi_2, F^{(3)}),\ \cdots M = (\Phi_{k-1} F^{(k)}).$$

We shall say that M is *composed* of the functions $F^{(1)}$, $F^{(2)}$, $\ldots F^{(k)}$ and we shall call the operation denoted by (2) the *composition* of the functions $F^{(1)}, F^{(2)}, \cdots F^{(k)}$. The operation of *composition* of the functions $F^{(s)}$ evidently possesses the *associative* property. Inversion of the elements of M can only produce changes in sign in the result.

The $F^{(s)}$ will be spoken of as the *divisors* of M. If M has no other divisors but itself, it will be spoken of as *prime*. If two functions have no *divisor* in common, they will be said to be *mutually prime*.

4. Without stopping to develop the theory of divisibility in the present sense, we can give directly a few of its proper-

ties and apply them to the results of the preceding sections. Thus, every regular function, which is not prime, can be decomposed into prime divisors, and this decomposition can be effected in more than one way. If a function divides one of the divisors of a function, it divides the function itself.

Two functions F and Φ will be isogenous when

$$F \equiv (\Psi, f), \quad \Phi = (\Psi, \phi),$$

where f, ϕ are point functions and f is a function of ϕ. If F and Φ are isogenous, so will be also the functions

$$(F, \Theta) \text{ and } (\Phi, \Theta).$$

No function is isogenous to a prime function; in order that a function may be found isogenous to a given function it is necessary and sufficient that the given function should admit a divisor which is a point function. That is, it is necessary for it to have the form $F \equiv (\Psi, f)$ with f a point function.

An elementary function is obtained by the composition of point functions, etc., etc.

12. *New considerations with reference to the relation of isogeneity*

1. So far we have been considering isogeneity between functions of hyperspaces of the same number of dimensions. We are now to generalize this relation so that it will apply to hyperspaces of different dimensions. Let us consider the two regular functions $\Phi|[S_r]|$, $\Psi|[S_t]|$, with $r > t$, and write

$$\frac{\partial \Phi}{\partial(x_{\iota_1} \cdots x_{\iota_{r+1}})} = a_{\iota_1 \cdots \iota_{r+1}}, \quad \frac{\partial \Psi}{\partial(x_{\iota_1} \cdots x_{\iota_{t+1}})} = b_{\iota_1 \cdots \iota_{t+1}}.$$

We shall say that Φ and Ψ are isogenous when the following conditions are satisfied:

$$(1) \qquad \sum_{1}^{r+2}{}_{s} (-1)^s a_{\iota_1 \cdots \iota_{s-1} \iota_{s+1} \cdots \iota_{r+2}} b_{\iota_s h_1 \cdots h_t} = 0.$$

In the case where r is equal to t, these equations imply that the functions not only are isogenous in our first sense, but also that they are elementary. Conversely, if two elementary functions of hyperspaces of the same number of dimensions are isogenous in the sense of section 6, they are also in the present sense.

2. It is easy to show that *every function which admits Φ as divisor is isogenous to Ψ.* In fact, if we take

$$c_{i_1 \cdots i_{v+2}} = \sum (-1)^{\binom{h_1 \cdots h_{v+2}}{i_1 \cdots i_{v+2}}} a_{h_1 \cdots h_{r+1}} a'_{r+2, \cdots v+2},$$

we shall have $$\sum_{1}^{v+3}{}_{s} (-1)^s c_{i_1 \cdots i_{s-1} i_{s+1} \cdots i_{v+3}} b_{i_s h_1 \cdots h_t} = 0,$$

which proves the theorem.

3. We can now generalize a theorem given in section 7, article 2. We have:

The necessary and sufficient condition that $\phi \mid [S_r] \mid$ shall be isogenous to the elementary function $\Psi \mid [S_{r-t}] \mid$, is that

$$\Phi \mid [S_r] \mid = (\Psi, \Theta). \tag{2}$$

That the condition is sufficient can be shown without any difficulty. In order to show that it is also necessary, let us write

$$\frac{d\Phi}{d(x_{i_1} \cdots x_{i_{r+1}})} = a_{i_1 \cdots i_{r+1}}, \quad \frac{d\Psi}{d(x_{i_1} \cdots x_{i_{r-t+1}})} = b_{i_1 \cdots i_{r-t+1}},$$

$$\frac{d\Theta}{d(x_{i_1} \cdots x_{i_t})} = c_{i_1 \cdots i_t},$$

$$b_{i_1 \cdots i_{r-t+1}} = \frac{d(\phi_1, \phi_2 \cdots \phi_{r-t+1})}{d(x_{i_1} x_{i_2} \cdots x_{i_{r-t+1}})}.$$

We shall show that if (1) is true, (2) is also true; that is, that

$$a_{i_1 \cdots i_{r+1}} = \sum (-1)^{\binom{h_1 \cdots h_{r+1}}{i_1 \cdots i_{r+1}}} b_{h_1 \cdots h_{r-t+1}} c_{h_{r-t+2} \cdots h_{r+1}}. \tag{2'}$$

For this purpose let us make a change of variable, taking instead of $x_1, x_2, \cdots x_n$ the new variables $\phi_1, \phi_2, \cdots \phi_{t+1}, x_{t+2}, \cdots x_n$. If we indicate with a prime the symbols that belong with the new variables, we shall have

(i) If $h_{r-t+2}, h_{r-t+3}, \cdots h_r \neq \phi_1, \phi_2, \cdots \phi_{r-t+1}$, then

$$c_{h_{r-t+2} \cdots h_r} = c'_{h_{r-t+2} \cdots h_r}$$

$$+\sum(-1)^{\binom{h_{p_1} \cdots h_{p_t}}{h_{r-t+2} \cdots h_{r+1}}} c'_{h_{p_1} \cdots h_{p_\nu} l_1 \cdots l_s} \frac{d(\phi_{l_1} \cdots \phi_{l_s})}{d\left(x_{h_{p_{\nu+1}}} \cdots x_{h_{p_t - 1}}\right)},$$

in which $l_1, \cdots l_s$ are s of the numbers $1, 2, \cdots r-t+1$, and $h_{p_1} \cdots h_{p_t}$ is a permutation of the numbers $h_{r-t+2}, \cdots h_{r+1}$.

(ii) If one of the numbers $h_{r-t+2} \cdots h_r$ is equal to one of the numbers $1, 2, \cdots t+1$, then

$$c_{h_{r-t+2} \cdots h_r} = \sum(-1)^{\binom{h_{p_1} \cdots h_{p_{t-1}}}{h_{r-t+2} \cdots h_r}} c'_{h_{p_1} \cdots h_{p_\nu} l_1 \cdots l_s} \frac{d(\phi_{l_1} \cdots \phi_{l_s})}{d(x_{h_{p_{\nu+1}}} \cdots x_{h_{p_{t-1}}})}$$

Equation (2′) will then become

$$(2'') \quad a_{i_1 \cdots i_{r+1}} = \sum(-1)^{\binom{h_1 \cdots h_{r+1}}{i_1 \cdots i_{r+1}}} b_{h_1 \cdots h_{r-t+1}} c'_{h_{r-t+2} \cdots h_{r+1}}$$

$$+\sum(-1)^{\binom{h_1 \cdots h_{r+1}}{i_1 \cdots i_{r+1}}} \frac{d(\phi_1 \cdots \phi_{r-t+1})}{d(x_{h_1} \cdots x_{h_{r-t+1}})}$$

$$\sum(-1)^{\binom{h_{p_1} \cdots h_{p_t}}{h_{r-t+2} \cdots h_{r+1}}} c'_{h_{p_1} \cdots h_{p_\nu} l_1 \cdots l_s} \frac{d(\phi_{l_1} \cdots \phi_{l_s})}{d(x_{h_{p_{\nu+1}}} \cdots x_{h_{p_t}})},$$

in which the first sum is extended over all the possible combinations of the indices $h_{r-t+2} \cdots h_{r+1}$ which do not contain any of the numbers $1, 2, \cdots r-t+1$. The second sum may be rewritten in the form

$$\sum(-1)^{\binom{h_{p_1} \cdots h_{p_{r+1}}}{i_1 \cdots i_{r+1}}} c'_{h_{p_1} \cdots h_{p_\nu} l_1 \cdots l_s}$$

$$\sum(-1)^{h_{p_t}}$$

$$\frac{d(\phi_1 \cdots \phi_{r-t+1})}{d(x_{h_1} \cdots x_{h_{r-t+1}})} \frac{d(\phi_{l_1} \cdots \phi_{l_s})}{d(x_{h_{p_{\nu+1}}} \cdots x_{h_{p_{t-1}}})},$$

whence it vanishes. The equation (2″) reduces then to

$$(2''') \qquad a_{i_1\cdots i_{r+1}} = \sum (-1)^{\binom{h_1\cdots h_{r+1}}{i_1\cdots i_{r+1}}} b_{h_1\cdots h_{r-t+1}} c'_{h_{r-t+2}\cdots h_{r+1}}.$$

In particular we have

$$a_{1,2,\cdots t+1, i_{t+2}\cdots i_{r+1}} = b_{1,2,\cdots t+1} c'_{i_{t+2}\cdots i_{r+1}}$$

so that

$$(3) \qquad c'_{i_{t+2}\cdots i_{r+1}} = \frac{a_{1,2,\cdots t+1, i_{t+2}\cdots i_{r+1}}}{\left\{ \dfrac{d(\phi_1 \cdots \phi_{t+1})}{d(x_1 \cdots x_{i_{t+1}})} \right.}.$$

Now by following a process analogous to that of section 7, article 2, it is easy to show that all the equations (2‴) are a consequence of these last equations (3). And so it is sufficient for us to show that the quantities c', obtained from (3), satisfy the conditions of integrability. We have in fact

$$a'_{1,\cdots t+1, i_{t+2},\cdots i_{r+1}} = \frac{a_{1,\cdots t+1, i_{t+2},\cdots i_{r+1}}}{\dfrac{d(\phi_1 \cdots \phi_{t+1})}{d(x_1 \cdots x_{t+1})}},$$

while a' will be zero if it has less than $t+1$ of its subscripts taken from the numbers 1, 2, $\cdots t+1$. If we apply then a process of reasoning analogous to that of section 7, article 2, we find that the conditions of integrability will be satisfied for the quantities c'.

13. *Differentiation and integration*

1. If two functions $F \mid [S_n] \mid$, $\Phi \mid [S_r] \mid$ are regular and isogenous, we know that the ratio

$$\phi = \frac{\left(\dfrac{d\Phi}{dS_{r+1}}\right)}{\left(\dfrac{dF}{dS_{r+1}}\right)}$$

will be independent of the hyperspace S_{r+1}, and will depend merely upon the point of the space at which the derivative is taken. The quantity ϕ will then be a point function for the total space of n dimensions. We shall denote it with the symbol $\frac{d\Phi}{dF}$ and call it the *derivative of* Φ *with respect to* F.

As a fundamental theorem it can be shown that *the derivative of* Φ *with respect to* F *is isogenous to both of the functions* Φ *and* F. The proof of this theorem comes immediately from formula (1) of section 7, with reference to the definition given in the preceding section.

2. Consider now a point function f isogenous to a regular function $F \mid [S_r] \mid$. By fixing the direction of the hyperspace S_{r+1} (see section 1, article 2) the quantity $\frac{dF}{dS_{r+1}}$ will be defined (see section 3, article 7), and hence the quantity

$$\int_{S_{r+1}} f \frac{dF}{dS_{r+1}} dS_{r+1}$$

will also be defined. This integral we shall represent by the symbol

$$\int_{S_{r+1}} f dF.$$

Changing the direction of the hyperspace will change the sign of the integral.

We shall suppose that the hyperspace S_{r+1} is closed and forms the boundary of a hyperspace S_{r+2} immersed in a portion of the total hyperspace S_n throughout which f and F have no singularities. It follows that

$$\int_{S_{r+1}} f dF = \int_{S_{r+1}} f \sum \frac{dF}{d(x_{t_1} \cdots x_{t_{r+1}})} \alpha_{t_1 \ldots t_{r+1}} dS_{r+1}$$

$$= \int_{S_{r+1}} f \sum p_{t_1 \ldots t_{r+1}} \alpha_{t_1 \ldots t_{r+1}} dS_{r+1},$$

where the $\alpha_{i_1 \cdots i_{r+1}}$ are the direction cosines of the hyperspace S_{r+1}. If we choose properly the direction of the hyperspace S_{r+2} and apply the generalization of Stokes's theorem (see section 4) we shall have

$$\int_{S_{r+1}} f dF = \int_{S_{r+2}} \sum \beta_{i_1 \cdots i_{r+2}} \sum (-1)^{s-1} \frac{\partial (f p_{i_1 \cdots i_{s-1} i_{s+1} \cdots i_{r+2}})}{\partial x_{i_s}} dS_{r+2}$$

$$= \int_{S_{r+2}} \sum \beta_{i_1 \cdots i_{r+2}} \left\{ \sum (-1)^s p_{i_1 \cdots i_{s-1} i_{s+1} \cdots i_{r+2}} \frac{\partial f}{\partial x_{i_s}} \right.$$

$$\left. + f \sum (-1)^{s-1} \frac{\partial p_{i_1 \cdots i_{s-1} i_{s+1} \cdots i_{r+2}}}{\partial x_{i_s}} \right\} dS_{r+2} = 0.$$

Hence we have the theorem expressed by the formula

$$(1) \qquad \int_{S_{r+1}} f dF = 0.$$

If, instead of a single hyperspace S_{r+1} we have the hyperspaces $S_{r+1}^{(i)}$ $(i = 1, 2, \cdots n)$ which bound a space S_{r+2} within which there are no singularities for f or F, we shall have the formula:

$$(1') \qquad \sum_1^n {}_i \int_{S_{r+1}^{(i)}} f dF = 0,$$

in which the directions of the hyperspaces $S_{r+1}^{(i)}$ are all to be chosen with reference to the conventions adopted for the generalization of Stokes's theorem.

The theorem enunciated in the formulæ (1) *and* (1′) *is the direct extension of Cauchy's theorem.*

3. Let us take away from the total hyperspace all those portions in which either f or F have singularities, and then introduce cuts in such a way that every closed hyperspace S_{r+1} may be taken as the complete boundary of a hyperspace S_{r+2}.

Take two hyperspaces S_r^0, S_r' such that a hyperspace S_{r+1} can be drawn to have them for its boundary, and choose the positive direction of S_r^0 and the negative direction of S_r' so as to correspond by the theorem of Stokes to one direction of the hyperspace S_{r+1}. With the direction of S_{r+1} fixed in this way, the integral

$$\int_{S_{r+1}} f dF \tag{2}$$

will be determined.

It is easy to show that the value of the integral (2) will not depend on the hyperspace S_{r+1}, but merely on S_r^0 and S_r. In fact if S_{r+1}' is another hyperspace which has these same two spaces for its boundary, the totality of S_{r+1} and S_{r+1}' will form a closed hyperspace, and from the hypotheses that we have made, we shall have

$$\int_{S_{r+1}+S_{r+1}'} f dF = 0,$$

from which the desired property follows.

Therefore the integral (2) can be indicated by the expression

$$\int_{S_r^0}^{S_r'} f dF. \tag{2'}$$

By changing the direction of S_{r+1} we change the sign of the integral; hence we may write

$$\int_{S_r^0}^{S_r'} f dF = -\int_{S_r'}^{S_r^0} f dF. \tag{3}$$

4. If we keep fixed the hyperspace S_r^0 and vary S_r', the integral (2′) may be regarded as a function (regular) of S_r', and we can write

$$\int_{S_r^0}^{S_r'} f dF = \Phi | [S_r'] |. \tag{4}$$

The function Φ *will be isogenous to* F *and we shall have*

$$\frac{d\Phi}{dF}=f, \tag{5}$$

that is to say, *the two operations of integration and differentiation are mutually inverse.*

14. *Isogeneity of order* r

1. A system of elementary functions will be said to have isogeneity of order r when all the functions of order greater than or equal to r, which are obtained from the system by means of composition (see section 11), vanish, while there is at least one function of order $r-1$ which does not vanish. All the elementary functions $\Phi|[S_t]|$ of the system must depend on certain functions $\phi_1, \phi_2, \cdots \phi_k, \cdots$ in such a way (see section 10) that

$$\frac{\partial\Phi}{\partial(x_{i_1}\cdots x_{i_{t+1}})}=\frac{d(\phi_{l_1}\cdots\phi_{l_{t+1}})}{d(x_{i_1}\cdots x_{i_{t+1}})},\ \Phi\equiv(\phi_{l_1}, \phi_{l_2}, \cdots \phi_{l_{t+1}}).$$

2. We have immediately the following theorems:

The necessary and sufficient condition for isogeneity of order r *that is*

$$\frac{d(\phi_{l_1}\cdots\phi_{l_{r+1}})}{d(x_{i_1}\cdots x_{i_{r+1}})}=0 \tag{1}$$

for every possible combination of the numbers $l_1, \cdots l_{r+1}, i_1, \cdots i_{r+1}$.

A function of order $r-1$ *is always isogenous to any other function of the system.*

In fact from (1) it follows that every function of order $r-1$ is isogenous to the functions of order zero of the system, that is, to the functions ϕ_s. We shall have then

$$q_{i_s h_1 \cdots h_t} = \frac{\partial \Phi}{\partial(x_{i_s} x_{h_1} \cdots x_{h_t})} = \sum_1^{t+1}{}_h (-1)^{u-1} \frac{\partial \phi_{l_u}}{\partial x_{i_s}} \frac{d(\phi_{l_1} \cdots \phi_{l_{u-1}} \phi_{l_{u+1}} \phi_{l_{t+1}})}{d(x_{h_1} \cdots x_{h_t})}$$

$$= \sum_1^{t+1} (-1)^{u-1} \frac{\partial \phi_{l_u}}{\partial x_{i_s}} N_u.$$

And if we let $\psi \mid [S_{r-1}] \mid$ represent one of the functions of order $r - 1$ of the system, and write

$$\frac{\partial \psi}{\partial(x_{i_1} \cdots x_{i_r})} = p_{i_1 \cdots i_r},$$

we shall have

$$\sum_1^{r+1}{}_s (-1)^s p_{i_1 \cdots i_{s-1} i_{s+1} \cdots i_{r+1}} q_{i_s h_1 \cdots h_t}$$

$$= \sum_1^{t+1} (-1)^{u-1} N_u \sum_1^{r+1}{}_s (-1)^s p_{i_1 \cdots i_{s-1} i_{s+1} \cdots i_{r+1}} \frac{\partial \phi_{l_u}}{\partial x_{i_s}}.$$

Every function of order $r - 1$ admits as divisor another function of the system of lower order (see section II, article 3).

3. Let us consider specially the functions of the system of order zero; that is, the functions $\phi_1, \phi_2, \cdots \phi_k, \cdots$. By means of the equations (1) we know that *there must be* r *of them, $\phi_1, \phi_2, \cdots \phi_r$, independent, of which all the others are functions*, and conversely, that *every function of $\phi_1, \phi_2, \cdots \phi_r$ will be an elementary function in the system, and will be of order zero.*

If we take two functions Φ and F of order $r - 1$, they will be isogenous, and we shall have the relation

$$\frac{d\Phi}{dF} = \phi(\phi_1, \phi_2, \cdots \phi_r). \tag{2}$$

Further, if we take an arbitrary function ϕ of order zero, that is a function of $\phi_1, \phi_2, \cdots \phi_r$, we shall have

$$\int_{S_r} \phi \, dF = 0, \tag{3}$$

where S_r is the complete boundary of a space S_{r+1} within which ϕ and F have no singularities. If we have

$$F \equiv (\phi_1, \phi_2, \cdots \phi_r),$$

then (3) can be written in the form

$$\int_{S_r} \phi \frac{d(\phi_1, \phi_2, \cdots \phi_r)}{d(\omega_1, \omega_2, \cdots \omega_r)} d\omega_1, d\omega_2, \cdots d\omega_r = 0,$$

$\omega_1, \cdots \omega_r$ being the parameters of the hyperspace S_r (see section 1, articles 1, 2). If we take

$$\frac{d\phi_t}{d\omega_s} d\omega_s = d_s\phi_t,$$

we shall have $$\int_{S_r} \phi \begin{vmatrix} d_1\phi_1 & d_2\phi_1 \cdots d_r\phi_1 \\ d_1\phi_2 & d_2\phi_2 \cdots d_r\phi_2 \\ . \quad . & . \quad . \quad . \quad . \\ d_1\phi_r & d_2\phi_r \cdots d_r\phi_r \end{vmatrix} = 0,$$

which is but a *generalization of Cauchy's theorem* (see the preceding section) put in a different form for the case of the elementary functions.

If S_r is not closed, but is bounded by two hyperspaces S^0_{r-1} and S_{r-1}, of which the first is fixed and the second variable, we shall have defined the expression

$$\Phi |[S_{r-1}]| = \int_{S^0_{r-1}}^{S_{r-1}} \phi \begin{vmatrix} d_1\phi_1 \cdots d_r\phi_1 \\ . \quad . \quad . \quad . \\ . \quad . \quad . \quad . \\ d_1\phi_r \cdots d_r\phi_r \end{vmatrix}.$$

Third Lecture

ON THE THEORY OF WAVES AND GREEN'S METHOD*

SECTION 1

LET a homogeneous liquid be subjected to certain forces and let it occupy a domain S. Let this domain be limited by a frontier σ which is composed partly of a set ω' of rigid boundaries, and partly of a free surface ω, where the pressure is P.

Let us suppose that the state of equilibrium is stable. We shall study the small oscillations of the fluid when it is displaced from the state of equilibrium.

The hydrodynamical equations of Lagrange are

$$\left.\begin{aligned}
\frac{d^2x}{dt^2}\cdot\frac{\partial x}{\partial x_0}+\frac{d^2y}{dt^2}\cdot\frac{\partial y}{\partial x_0}+\frac{d^2z}{dt^2}\cdot\frac{\partial z}{\partial x_0}&=\frac{\partial}{\partial x_0}\left(V-\frac{P}{\rho}\right)\\
\frac{d^2x}{dt^2}\cdot\frac{\partial x}{\partial y_0}+\frac{d^2y}{dt^2}\cdot\frac{\partial y}{\partial y_0}+\frac{d^2z}{dt^2}\cdot\frac{\partial z}{\partial y_0}&=\frac{\partial}{\partial y_0}\left(V-\frac{P}{\rho}\right)\\
\frac{d^2x}{dt^2}\cdot\frac{\partial x}{\partial z_0}+\frac{d^2y}{dt^2}\cdot\frac{\partial y}{\partial z_0}+\frac{d^2z}{dt^2}\cdot\frac{\partial z}{\partial z_0}&=\frac{\partial}{\partial z_0}\left(V-\frac{P}{\rho}\right)
\end{aligned}\right\}\qquad(1)$$

where x, y, z, denote the coördinates of points of the fluid at time t, x_0, y_0, z_0 the initial coördinates, V the potential function, P the pressure, ρ the density.

2. Let x_0, y_0, z_0 be the coördinates which correspond to the state of stable equilibrium, ξ, η, ζ the components of displacement of each particle with respect to its position of equilibrium.

Then $\qquad x=x_0+\xi, \qquad y=y_0+\eta, \qquad z=z_0+\zeta.$

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* Translated from the French by Professor Percy John Daniell, of the Rice Institute.

If we consider the displacements as infinitesimals of the first order and if we neglect terms of order higher than the first, the equations (1) become

$$\left.\begin{aligned}\frac{d^2\xi}{dt^2}&=\frac{\partial}{\partial x_0}\left(V-\frac{P}{\rho}\right)\\ \frac{d^2\eta}{dt^2}&=\frac{\partial}{\partial y_0}\left(V-\frac{P}{\rho}\right)\\ \frac{d^2\zeta}{dt^2}&=\frac{\partial}{\partial z_0}\left(V-\frac{P}{\rho}\right)\end{aligned}\right\}.$$

For simplification the indices 0 are suppressed and x, y, z denote the coördinates of each particle in the position of equilibrium.

Then

$$\left.\begin{aligned}\frac{d^2\xi}{dt^2}&=\frac{\partial}{\partial x}\left(V-\frac{P}{\rho}\right)\\ \frac{d^2\eta}{dt^2}&=\frac{\partial}{\partial y}\left(V-\frac{P}{\rho}\right)\\ \frac{d^2\zeta}{dt^2}&=\frac{\partial}{\partial z}\left(V-\frac{P}{\rho}\right)\end{aligned}\right\}. \qquad (2)$$

The condition of incompressibility can be written as

$$\frac{\partial\xi}{\partial x}+\frac{\partial\eta}{\partial y}+\frac{\partial\zeta}{\partial z}=0. \qquad (3)$$

On account of (2) we can put

$$\xi=\frac{\partial\Phi}{\partial x}, \qquad \eta=\frac{\partial\Phi}{\partial y}, \qquad \zeta=\frac{\partial\Phi}{\partial z},$$

Φ being the potential of displacement.

Then the equations (2) become

$$\frac{d^2\Phi}{dt^2}-V+\frac{P}{\rho}=c, \qquad (4)$$

where c is constant with respect to x, y, z, but may vary with t.

The equation (3) becomes

$$\Delta^2\Phi = 0.$$

At points of the liquid where it touches the rigid boundary

$$\xi \cos nx + \eta \cos ny + \zeta \cos nz = 0,$$

if n denotes the normal to the boundary.

This condition becomes $\frac{\partial\Phi}{\partial n} = 0.$

3. Let us return to the equation (4). If we put

$$V - \frac{P}{\rho} + c = H,$$

the equation (4) becomes $\frac{d^2\Phi}{dt^2} = H.$ (4′)

The free surface of the fluid has been denoted by ω. Let us suppose that the potential function V and the pressure P, which correspond to each particle of fluid belonging to ω are functions of the coördinates of the point occupied by the particle independently of the form of the liquid. If this hypothesis is not correct, since the displacements are infinitesimal, we can neglect the variations produced by the changes in form of the fluid so that we can always proceed as if the hypothesis were correct.

In the state of equilibrium H is constant on ω. Therefore the equation of this surface will be

$$H = H_0 = \text{constant}.$$

Let us now calculate H when a point of the surface ω is displaced when ξ, η, ζ are the components of displacement.

If we neglect infinitesimals of a higher order than the first,

$$H = H_0 + \frac{\partial H}{\partial x}\xi + \frac{\partial H}{\partial y}\eta + \frac{\partial H}{\partial z}\zeta.$$

Then putting $\lambda^2 = \left(\frac{\partial H}{\partial x}\right)^2 + \left(\frac{\partial H}{\partial y}\right)^2 + \left(\frac{\partial H}{\partial z}\right)^2$,

$$\frac{\partial H}{\partial x} = \lambda \cos nx, \quad \frac{\partial H}{\partial y} = \lambda \cos ny, \quad \frac{\partial H}{\partial z} = \lambda \cos nz, \tag{5}$$

when n is the normal to the surface ω.

Then
$$H = H_0 + \lambda\,(\xi \cos nx + \eta \cos ny + \zeta \cos nz)$$
$$= H_0 + \lambda \frac{\partial \Phi}{\partial n};$$

combining this with equation (4′)

$$\frac{\partial^2 \Phi}{\partial t^2} = H_0 + \lambda \frac{\partial \Phi}{\partial n}$$

or

$$\frac{\partial^2 \Phi}{\partial t^2} = \lambda \frac{\partial \Phi}{\partial n},$$

since Φ is determinate except for a quantity which is constant with respect to the time.

Let us take the normal n as directed toward the interior of the fluid, and let us suppose that $V - \frac{P}{\rho}$ increases on moving ω and following the positive direction of n.

Then when n is positive, $\frac{\partial H}{\partial n} > 0$,

or by virtue of the equations (5)

$$\frac{\partial H}{\partial n} = \frac{\partial H}{\partial x} \cos nx + \frac{\partial H}{\partial y} \cos ny + \frac{\partial H}{\partial z} \cos nz = \lambda,$$

it follows that $\lambda > 0$.

The problem of waves can be presented in the following manner.

4. To determine a function Φ regular within the domain S which satisfies the equation

$$(A) \qquad \Delta^2 \Phi = 0$$

within S and which in the part ω' of the boundary satisfies the condition

$$(B)\qquad \frac{\partial\Phi}{\partial n}=0$$

and in the part ω satisfies the condition

$$(C)\qquad \frac{\partial^2\Phi}{\partial t^2}=\lambda\frac{\partial\Phi}{\partial n},$$

where λ is a positive quantity independent of the time, and n is the normal to the boundary directed toward the interior of the domain S.

SECTION 2

1. We can make a comparison between the problem we are about to consider and that of the vibrations of elastic media, and other problems of mathematical physics. The problem of the vibrations of elastic media is based upon the equation

$$\frac{\partial^2 u}{\partial t^2}=\alpha^2\Delta^2 u. \qquad (6)$$

The problem of the propagation of heat in the case of varying temperature leads to the equation

$$\frac{\partial V}{\partial t}=a\Delta^2 V. \qquad (7)$$

The problems of potential and of stationary temperatures in isotropic bodies depend upon the equation of Laplace

$$\Delta^2 W=0. \qquad (8)$$

These three equations are respectively of *hyperbolic*, *parabolic*, and *elliptic* types.

The question we have considered in section 1 belongs to the elliptic type on account of the equation (A) of section 1, which is the equation of Laplace; but it is the condition which must be satisfied on the surface ω of the boundary which leads to the essential difference between this problem and the problems of potential and stationary temperatures. In fact, in the problems of potential the conditions at the boundary are reduced to that of giving the values of the unknown function or of its normal derivative; in those of stationary temperatures a linear relation between the unknown function and its normal derivative is known. But in the case of waves the condition at the boundary (equation (C) of section 1) introduces a new variable, the time, which makes the problem one of four variables. In respect to the number of variables the problem of waves is similar to the problems of vibrations and varying temperatures. It differs from them, however, because equations (6) and (7) have real characteristics. There are no real characteristics in the problem of the waves of liquids. We shall give a theorem in section 3 which will show the difference, from a physical standpoint, between waves in elastic media and waves in liquids.

2. There are two general methods in which the different problems we are investigating can be treated.

That of the separation of variables consists in separating the time from the space variables.

Let us put in the equation (6)

$$U = \sin mt \cdot u(x, y, z), \tag{9}$$

where m is a constant.

The equation becomes

$$m^2 u + \alpha^2 \Delta^2 u = 0, \tag{10}$$

where the time has disappeared. If, for example, on the boundary $U = 0$, u must be taken $= 0$ there. We are led to find values of m for which the previous equation has solutions which are not identically zero (special solutions). The general solution is obtained by forming an infinite series of solutions of the form (9) multiplied by arbitrary constants of such values that U and $\frac{\partial U}{\partial t}$ for $t = 0$ have the values of the given functions of x, y, z.

The question of determining the special solutions has been resolved by Poincaré; the theory of integral equations has been used and Mr. Hilbert, Mr. Schmidt, and others have founded the theory of series of special solutions.

Similarly an analogous process can be employed for equation (7) if we put $V = e^{mt}v(x, y, z)$; that is to say, by separating the time from the variables x, y, z.

Equation (7) reduces then to

$$mv + a\Delta^2 v = 0,$$

which is exactly analogous to equation (10).

3. The same method of the separation of the variables can be applied to the problem of waves in liquids.

If we put $\Phi = \sin mt\, \phi(x, y, z)$ equation (A) of section 1 becomes

$$\Delta^2\phi = 0,$$

equation (B) is

$$\frac{\partial \phi}{\partial u} = 0,$$

and equation (C) must be replaced by

$$m^2\phi + \lambda\frac{\partial \phi}{\partial n} = 0.$$

Here again the values of m corresponding to solutions ϕ which are not identically zero (special solutions) must be found.

By series of special solutions the general solution can be obtained. To calculate the values of m the method of Poincaré with those of integral equations can be used.

4. But we wish to set aside the process of the separation of variables and to pass on to the other general method. It

is the method which is connected with the ideas which Green used for the first time for the equation of Laplace and which, little by little, has been also used for other types of equations. By this point of view Kirchhoff arrived at his celebrated formula which expresses the principle of Huyghens. He applied Green's method to equation (6).

Betti has also applied an analogous method to equation (7).

We wish to show that a general formula can be found in the case of waves of fluids of a type which presents some analogies to these formulæ. I have had occasion to mention this formula without giving any development from it in my lectures at Stockholm. We shall now develop it and demonstrate in detail some applications of it.

SECTION 3

1. We shall begin by demonstrating in this paragraph some general theorems.

First Theorem. If Φ *is the function which satisfies the conditions* (*A*), (*B*), (*C*) *of section* 1, *it is determinate if the values* Φ_0, $\left(\frac{\partial \Phi}{\partial t}\right)_0$ *of* Φ *and* $\left(\frac{\partial \Phi}{\partial t}\right)$ *for* $t = 0$ *on the surface* ω *are known.*

Demonstration. Let Φ_1, Φ_2 be two functions which satisfy the conditions to which Φ is subjected.

Their difference $\Phi_3 = \Phi_1 - \Phi_2$ also satisfies the equations (*A*), (*B*), (*C*) and further we have

$$(\Phi_3)_0 = 0 \qquad \left(\frac{\partial \Phi_3}{\partial t}\right)_0 = 0$$

for $t = 0$ on the surface ω.

Let us now calculate

$$\Omega = \frac{1}{2}\frac{\partial}{\partial t}\int_\omega \frac{1}{\lambda}\left(\frac{\partial \Phi_3}{\partial t}\right)^2 d\omega.$$

On account of equation (C) we shall have

$$\Omega = \int_\omega \frac{1}{\lambda}\left(\frac{\partial \Phi_3}{\partial t}\right)\left(\frac{\partial^2 \Phi_3}{\partial t^2}\right) d\omega = \int_\omega \left(\frac{\partial \Phi_3}{\partial t}\right)\left(\frac{\partial \Phi_3}{\partial n}\right) d\omega.$$

But on ω' $\frac{\partial \Phi_3}{\partial n} = 0$ and therefore

$$\Omega = \int_\sigma \left(\frac{\partial \Phi_3}{\partial t}\right)\left(\frac{\partial \Phi_3}{\partial n}\right) \partial\sigma.$$

Applying a well-known transformation,

$$-\Omega = \int_S \left(\frac{\partial}{\partial x}\frac{\partial \Phi_3}{\partial t} \cdot \frac{\partial \Phi_3}{\partial x} + \frac{\partial}{\partial y}\frac{\partial \Phi_3}{\partial t} \cdot \frac{\partial \Phi_3}{\partial y} + \frac{\partial}{\partial z}\frac{\partial \Phi_3}{\partial t} \cdot \frac{\partial \Phi_3}{\partial z}\right) \partial S$$
$$+ \int_S \frac{\partial \Phi_3}{\partial t} \Delta^2 \Phi_3 dS.$$

The third term $= 0$; then

$$-\Omega = \frac{1}{2}\frac{\partial}{\partial t}\int_S \left\{\left(\frac{\partial \Phi_3}{\partial x}\right)^2 + \left(\frac{\partial \Phi_3}{\partial y}\right)^2 + \left(\frac{\partial \Phi_3}{\partial z}\right)\right\} dS$$

and it follows that

$$\frac{1}{2}\frac{\partial}{\partial t}\left[\int_\omega \frac{1}{\lambda}\left(\frac{\partial \Phi_3}{\partial t}\right)^2 d\omega + \int_S \left\{\left(\frac{\partial \Phi_3}{\partial x}\right)^2 + \left(\frac{\partial \Phi_3}{\partial y}\right)^2 + \left(\frac{\partial \Phi_3}{\partial z}\right)^2\right\} dS\right] = 0.$$

Integrating with respect to the time,

$$\int_\omega \frac{1}{\lambda}\left(\frac{\partial \Phi_3}{\partial t}\right)^2 d\omega + \int_S \left\{\left(\frac{\partial \Phi_3}{\partial x}\right)^2 + \left(\frac{\partial \Phi_3}{\partial y}\right)^2 + \left(\frac{\partial \Phi_3}{\partial z}\right)^2\right\} dS = c, \qquad (11)$$

where c is constant with respect to the time.

Then if $(\Phi_3)_0 = 0$ for $t = 0$ on ω, since $\frac{\partial \Phi_3}{\partial n} = 0$ on ω' $(\Phi_3)_0$ must be zero in the domain S. Consequently, the second integral in the formula (11) will be 0 for $t = 0$. In the same way, since $\left(\frac{\partial \Phi_3}{\partial t}\right)_0 = 0$, the first integral will be 0 for $t = 0$. It follows that $c = 0$, and the conclusion can be drawn that Φ_3 will be 0 for every value of t and therefore $\Phi_1 = \Phi_2$. Q. E. D.

2. *Second Theorem. If at a certain instant the molecules belonging to a part of the domain S are not displaced from the position of equilibrium, any molecule of the fluid is not displaced from the position of equilibrium.*

Demonstration. If ξ, η, ζ are 0 in any part of S, Φ will be constant in this part, and since it is an harmonic function regular in S, it will be everywhere constant. Consequently ξ, η, ζ will be 0 at all points of S. Q. E. D.

Third Theorem. If at a certain instant the molecules belonging to a part of the domain S are not displaced from the position of equilibrium and have no velocity, the fluid will remain always in the position of equilibrium.

Demonstration. If ξ, η, ζ and $\frac{d\xi}{dt}$, $\frac{d\eta}{dt}$, $\frac{d\zeta}{dt}$ are 0 in one part of the domain S at a certain instant, Φ and $\frac{d\Phi}{dt}$ will be constant in this part and therefore they will be constant in the whole domain S at the same instant. By virtue of the first theorem they will be constant in S at every instant and consequently the liquid will have no motion. Q. E. D.

3. These propositions show us the essential difference which exists between waves in liquids and waves in elastic media. In elastic media the motion is propagated with a certain velocity from one part to another; in liquids the motion reaches the whole mass contemporaneously, at least when the fluid does not remain in a constant state of equilibrium. In the case of liquids there is no propagation of motion and consequently one cannot speak of the velocity of propagation.

SECTION 4

1. Let Φ and Ψ be two functions which satisfy the conditions (A), (B), (C) of section 1.

By virtue of Green's theorem

$$\int_{\sigma}\left(\phi\frac{\partial\psi}{\partial n}-\psi\frac{\partial\phi}{\partial n}\right)d\sigma=0$$

on account of (B) $\displaystyle\int_{\omega}\left(\phi\frac{\partial\psi}{\partial n}-\psi\frac{\partial\phi}{\partial n}\right)d\omega=0.$

Using (C) this becomes

$$\int_{\omega}\left(\Phi\left(\frac{\partial^2\psi}{\partial t^2}-\Psi\frac{\partial^2\phi}{\partial t^2}\right)\frac{1}{\lambda}d\omega=0. \qquad (12)$$

Let us now suppose that

$$\Psi=\frac{1}{r}+\chi,$$

where r denotes the distance between a point A (x_0, y_0, z_0) interior to the domain S and a point (x, y, z) and where χ is a regular function. Then the preceding formulæ are no longer valid for they presuppose that ψ is regular in the domain S. In this case formula (12) must be replaced by

$$4\pi\Phi_A+\int_{\bar{\omega}}\left(\Phi\frac{\partial^2\psi}{\partial t^2}-\psi\frac{\partial^2\phi}{\partial t^2}\right)\frac{1}{\lambda}d\omega=0, \qquad (12')$$

where Φ_A denotes the value of Φ at the point A.

Then $\displaystyle 4\pi\phi_A=-\frac{\partial}{\partial t}\int_{\bar{\omega}}\left(\phi\frac{\partial\psi}{\partial t}-\psi\frac{\partial\phi}{\partial t}\right)\frac{1}{\lambda}\partial\omega.$

Integrating between the limits 0 and t_1, we obtain

$$4\pi\int_0^{t_1}\phi_A dt=-\int_{\omega}\left\{\phi_1\left(\frac{\partial\psi}{\partial t}\right)_1-\psi_1\left(\frac{\partial\phi}{\partial t}\right)_1\right\}\frac{1}{\lambda}\partial\omega$$

$$+\int_{\omega}\left\{\phi_0\left(\frac{\partial\psi}{\partial t}\right)_0-\psi_0\left(\frac{\partial\psi}{\partial t}\right)_0\right\}\frac{1}{\lambda}\partial\omega$$

where $\phi_1\ \psi_1 \left(\frac{\partial\phi}{\partial t}\right)_1 \left(\frac{\partial\psi}{\partial t}\right)_1$ denote the functions ϕ, ψ and the derivatives $\frac{\partial\phi}{\partial t}\,\frac{\partial\psi}{\partial t}$ for $t=t_1$, while $\phi_0\ \psi_0 \left(\frac{\partial\phi}{\partial t}\right)_0 \left(\frac{\partial\psi}{\partial t}\right)_0$ denote the same quantities for $t=t_0$. Let us now suppose that ψ_1 and $\left(\frac{d\psi}{dt}\right)_1$ are 0 on ω.

Then

$$(D) \qquad \Phi(x_0,\, y_0,\, z_0\ t_1) = -\frac{1}{4\pi}\,\frac{d}{dt_1}\int_\omega \left\{\phi_0\left(\frac{\partial\psi}{\partial t}\right)_0 - \psi_0\left(\frac{\partial\phi}{\partial t}\right)_0\right\}\frac{1}{\lambda}\,d\omega.$$

The above formula gives us a knowledge of Φ at every point in S and for every value of t when the values of ϕ_0 $\left(\frac{\partial\phi}{\partial t}\right)_0$ are known on ω. (Compare with the first theorem of section 3.)

It is necessary to calculate further the function Ψ and consequently χ. This function plays, in this case, *a part which can be compared with that played by Green's function.*

It must be remarked that ψ_0 and $\left(\frac{d\psi}{dt}\right)_0$ should depend on t_1 since ψ_1 and $\left(\frac{d\psi}{dt}\right)_1$ should be 0. The variable t_1 appears then in the second member of the equation (D) because it is contained in ψ_0 and $\left(\frac{d\psi}{dt}\right)_0$.

SECTION 5

In this paragraph we shall give some applications of the fundamental formula (D) of the preceding paragraph. Let us suppose that S is a sphere of radius R and that ω is the surface of the sphere in such a way that there are no rigid boundaries.

Let us put

$$\psi = a_0 + \frac{(t_1-t)^2}{2\,!}a_2 + \frac{(t_1-t)^4}{4\,!}a_4 + \cdots,$$

$a_0, a_2, a_4 \cdots$ being coefficients independent of t_1 and t. We shall have

$$\psi_1 = a_0, \quad \left(\frac{d\psi}{dt}\right)_1 = 0.$$

But

$$\psi = \frac{1}{r} + \chi.$$

$$\therefore a_0 = \frac{1}{r_A} + (\chi)_1,$$

and since a_0 should be 0 on ω and χ should be a regular and harmonic function if we use the method of images we obtain

$$(\chi)_1 = -\frac{R}{l}\frac{1}{r_{A'}},$$

where A' denotes the image point of A with respect to the sphere, $r_{A'}$ is the distance of the point A' from the point (x, y, z), l is the distance from the center of the sphere to the point A.

Then

$$a_0 = \frac{1}{r_A} - \frac{R}{l}\frac{1}{r_{A'}}.$$

Let ρ be the radius vector, the pole being at the center of the sphere; then

$$\frac{\partial\psi}{\partial n} = -\frac{\partial\psi}{\partial\rho} = -\frac{\partial a_0}{\partial\rho} - \frac{(t_1 - t)^2}{2\,!}\frac{\partial a_2}{\partial\rho} - \frac{(t_1 - t)^4}{4\,!}\frac{\partial a_4}{\partial\rho}\cdots$$

$$\frac{\partial^2\psi}{\partial t^2} = a_2 + \frac{(t_1 - t)^2}{2\,!}a_4 + \cdots.$$

Consequently on the surface ω, *i.e.* for $\rho = R$

$$-\lambda\frac{\partial a_0}{\partial\rho} = a_2, \quad -\lambda\frac{\partial a_2}{\partial\rho} = a_4, \quad -\lambda\frac{\partial a_4}{\partial\rho} = a_6, \cdots.$$

Since a_0 is known, the regular harmonic functions $a_2, a_4, a_6 \cdots$ must be determinate when their values on the boundary of the sphere are known.

Let us begin by transforming the expression for a_0. Let us denote by γ the angle between the lines joining the center of the sphere to the points A and (x, y, z).

Then
$$a_0 = \frac{1}{(l^2 + \rho^2 - 2\, l\rho \cos\gamma)^{\frac{1}{2}}} - \frac{R}{l} \frac{1}{\left(\frac{R^4}{l^2} + \rho^2 - 2\frac{R^2}{l}\rho\cos\gamma\right)^{\frac{1}{2}}}$$

or
$$\frac{\rho}{R}\frac{\partial a_0}{\partial \rho} = \frac{\rho}{R}\frac{\partial}{\partial \rho}\left[\frac{1}{(l^2 + \rho^2 - 2\, l\rho\cos\gamma)^{\frac{1}{2}}}\right]$$
$$-\frac{\rho}{R}\frac{\partial}{\partial\rho}\left[\frac{R}{l} \cdot \frac{1}{\left(\frac{R^4}{l^2} + \rho^2 - 2\frac{R^2}{l}\rho\cos\gamma\right)^{\frac{1}{2}}}\right]$$

is a harmonic function which is equal to $\frac{\partial a_0}{\partial \rho}$ on the surface of the sphere; but it is not regular in the interior of the sphere. In fact, the first term of the second member becomes infinite for $\rho = l$, $\gamma = 0$. Then to calculate a_2 we cannot take the previous expression and multiply it by $-\lambda$ for a_2 must be regular in the interior of the sphere. But the following artifice may be used to calculate a_2.

Let us transform the first term of the second member by a transformation of reciprocal radii with respect to the sphere and let us multiply by $\frac{R}{\rho}$. The expression remains harmonic, possesses the same values on the boundary of the sphere, but becomes regular in the interior. To make the transformation of reciprocal radii it is sufficient to replace ρ by $\frac{R^2}{\rho}$. Thus the first term of the previous expression becomes
$$-R^2\frac{R^2 - l\rho\cos\gamma}{(l^2\rho^2 - R^4 - 2\, lR^2\rho\cos\gamma)^{\frac{3}{2}}}.$$

The second term equals
$$\frac{\rho l(l\rho - R^2\cos\gamma)}{(R^4 + l^2\rho^2 - 2\, lR^2\rho\cos\gamma)^{\frac{3}{2}}}$$

It is found then that

$$a_2 = -\lambda \frac{l^2\rho^2 - R^4}{(R^4 + l^2\rho^2 - 2\,l\rho R^2 \cos\gamma)^{\frac{3}{2}}}.$$

In calculating a_4, $a_6 \cdots$ there are no more difficulties and

$$a_4 = -\lambda^2 \frac{\rho}{R} \frac{\partial}{\partial\rho}\left[\frac{R^4 - l^2\rho^2}{R^4 + l^2\rho^2 - 2\,l\rho R^2 \cos\gamma)^{\frac{3}{2}}}\right]$$

$$= -\frac{\lambda^2}{R} \frac{\partial}{\partial \log \rho}\left[\frac{R^4 - l^2\rho^2}{(R^4 + l^2\rho^2 - 2\,l\rho R^2 \cos\gamma)^{\frac{3}{2}}}\right].$$

In general,

$$a_{2n} = (-1)^{n-1} \frac{\lambda^n}{R^{n-1}} \frac{\partial^{n-1}}{\partial(\log\rho)^{n-1}}\left[\frac{R^4 - l^2\rho^2}{R^4 + l^2\rho^2 - 2\,l\rho R^2 \cos\gamma)^{\frac{3}{2}}}\right].$$

Consequently,

$$\Psi = a_0 + \sum_1^\infty (-1)^{n-1} \frac{\lambda^n}{R^{n-1}} \frac{\partial^{n-1}}{\partial(\log\rho)^{n-1}}$$

$$\left[\frac{R^4 - l^2\rho^2}{R^4 + l^2\rho^2 - 2\,l\rho R^2 \cos\gamma)^{\frac{3}{2}}}\right] \frac{(t_1 - t)^{2n}}{2n\,!}.$$

$$\frac{\partial\Psi}{\partial t} = -\sum_1^\infty (-1)^{n-1} \frac{\lambda^n}{R^{n-1}} \frac{\partial^{n-1}}{\partial(\log\rho)^{n-1}}$$

$$\left[\frac{R^4 - l^2\rho^2}{(R^4 + l^2\rho^2 - 2\,l\rho R^2 \cos\gamma)^{\frac{3}{2}}}\right] \frac{(t_1 - t)^{2n-1}}{(2n-1)\,!}.$$

In order to calculate the formula (*D*) of section 4 it is necessary to evaluate ψ_0 and $\left(\frac{d\psi}{dt}\right)_0$, that is to say, to put $t = 0$ in the previous series. Further it is the values at the surface of the sphere which have to be found. Finally, this expression must be derived with respect to t_1.

Let us then adopt polar coördinates and put

$x = \rho \sin\theta \cos\phi, \quad y = \rho \sin\theta \sin\phi, \quad z = \rho\cos\theta,$

$x_0 = l \sin\theta_0 \cos\phi_0, \quad y_0 = l \sin\theta_0 \sin\phi_0, \quad z = l \cos\theta_0.$

Then $\cos\gamma = \cos\phi\cos\phi_0 + \sin\phi\sin\phi_0\cos(\theta-\theta_0)$.

Let us write

$$\Theta(l,\theta_0,\phi_0,\theta,\phi,t) = \sum_1^\infty (-1)^{n-1}\frac{\lambda^{n-1}}{R^{n-1}}\frac{\partial^{n-1}}{\partial(\log l)^{n-1}}\left[\frac{R^2-l^2}{R^2+l^2-2Rl\cos\gamma)^{\frac{3}{2}}}\right]\frac{t^{2n-1}}{(2n-1)!}.$$

Formula (*D*) can be written

$$(D_a)\Phi(l,\theta_0,\phi_0,t) = \frac{R}{\gamma\pi}\int_\omega \phi'_0(\theta,\phi)\Theta(l,\theta_0,\phi_0,\theta,\phi,t)\sin\theta d\theta d\phi$$

$$+\frac{R}{\gamma\pi}\frac{d}{dt}\int_\omega \phi_0(\theta,\phi)\Theta(l,\theta_0,\phi_0,\theta,\phi,t)\sin\theta d\theta d\phi,$$

where for simplification we have written

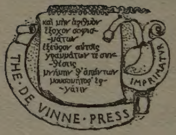
$$\Phi_0(\theta,\phi) = \phi_0(R,\theta,\phi,t),\ t=0$$

$$\Phi'_0(\theta,\phi) = \left\{\frac{d}{dt}\phi_0(R,\theta,\phi,t)\right\},\ t=0.$$

The formula we have been seeking to find is the general formula in the case of the sphere.

If, instead of a sphere, the liquid occupies a hemisphere and the diametral plane constitutes the rigid boundary so that the curved surface is free, the method of images will provide the solution in a similar manner. The same holds in the case where the liquid occupies a section of a sphere between two rigid diametral planes the angle between which equals $\frac{\pi}{n}$, where n is an integer.

VITO VOLTERRA.


καὶ μὴν ἀριθμὸν
ἔξοχον σοφισ-
-μάτων
ἐξεῦρον αὐτοῖς
γραμμάτων τε συν-
-θέσεις
μνήμην θ' ἁπάντων
μουσομήτορ' ἐρ-
-γάνην
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